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Compatibility in abstract Algebraic Structures

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Abstract

Compatible Hamiltonian pairs play a crucial role in the structure theory of integrable systems. In this paper we consider the question of how much of the structure given by compatibility is bound to the situation of hamiltonian dynamic systems and how much of that can be transferred to a complete abstract situation where the algebraic structures under consideration are given by bilinear maps on some module over a commutative ring. Under suitable modification of the corresponding definitions, it turns out that notions like, *compatible*, *hereditary*, *invariance* and *Virasoro algebra* may be transferred to the general abstract setup. Thus the same methods being so successful in the area of integrable systems, may be applied to generate suitable abelian algebras and hierarchies in very general algebraic structures.

1 Introduction

In her work on integrable systems, starting with the pioneering papers [6], [7], [8] and culminating in her account on Dirac structures [1] Irene Dorfman, not only paid special attention to those algebraic structures which allow the generation of abelian substructures, but also created some of the most powerful methods to generate dynamic systems having large abelian symmetry groups. In this context, also the paper [2], which in its ideas certainly is one of the crucial contributions for integrability in multidimensions, should be taken into account (compare [5] for an application of similar ideas).

By her work on compatible hamiltonian pairs Irene Dorfman strongly influenced the perspectives of the whole field. The ideas she shaped in the her early work, now infiltrate under a variety of different notions and methods, the whole field. For example, these ideas can be found in connection with *hereditary* or *Nijenhuis operators, Virasoro algebras* and *mastersymmetries.*

It seems a fundamental problem to check how far these ideas eventually may reach into other areas, in order to generate invariant structures in those fields which do not have access to the infinitesimal aspects which underly the study of dynamic systems. If that were the case, then one day methods similar to those known from integrability, maybe slightly modified, may be applied to *time discrete systems, automata, invariant substructures* of general algebras and other areas not yet targeted for the far reaching methods coming from the now solidly established area of integrable systems.

The present paper may be a small contribution towards enlarging our notions and methods to a wider area of application. How the results of this paper are applied in the classical situation of Hamiltonian systems see [4]. The main message of the results of the present paper is that *compatibility* is more a property of homomorphisms with respect to bilinear structures than a property connected to vector fields.

We start our considerations by defining hereditary structures, in more or less arbitrary algebraic structures, such that the crucial results about generating abelian substructures out of one or several invariants may be obtained. Then we show that a general notion of compatibility of homomorphisms in abstract algebraic structures may be characterized by this notion of hereditaryness. Thus the power of compatible hamiltonian structures is made available to a wider area of applications not needing the usual ingredients of the underlying structure of tensor bundles and Lie algebra modules.

Thereafter the notion of compatible deformations of products is introduced and the paper is concluded by introducing Virasoro algebras in general algebras and showing that these are just another aspect of the notions presented so far.

2 Hereditary structures and invariance in general algebras

Fix a commutative ring $\mathbf{F}$. Let $\mathcal{L}$ be some module over $\mathbf{F}$ and consider $(\mathcal{L}, \bullet)$, where $\bullet$ is some binary bilinear operator on $\mathcal{L}$. We call $(\mathcal{L}, \bullet)$ the **reference algebra**, $(\mathcal{L}, \bullet)$ is not necessarily an associative algebra. For short, binary bilinear operators in modules over $\mathbf{F}$ are called *products*. Two elements $a, b \in \mathcal{L}$ are said to *commute* if

$$a \bullet b = b \bullet a \ . \tag{2.1}$$

Let furthermore Λ be another module over $\mathbf{F}$ and consider a linear

$$\Theta : \Lambda \to \mathcal{L} \ . \tag{2.2}$$

from Λ into the reference algebra $\mathcal{L}$. We call a product $[\ ,\ ]$ in Λ a **Θ-product** if Θ is a homomorphism into $(\mathcal{L}, \bullet)$, i.e. if

$$\Theta[a, b] = (\Theta a) \bullet (\Theta b) \text{ for all } a, b \in \Lambda. \tag{2.3}$$

To emphasize that some product is a Θ-product we write $[\ ,\ ]_\Theta$ instead of $[\ ,\ ]$. Here, the symbols $[\ ,\ ]$ or $[\ ,\ ]_\Theta$ should not be confused with Lie algebras, also the algebra under consideration is not assumed to be antisymmetric. When we consider a antisymmetric algebras we shall write $[\![\ ,\ ]\!]$ instead.

A linear $\Phi : \mathcal{L} \to \mathcal{L}$ is said to be **hereditary** if

$$[a,b]_\Phi := (\Phi a) \bullet b + a \bullet (\Phi b) - \Phi(a \bullet b) \tag{2.4}$$

defines a Φ-product in $\mathcal{L}$. Recalling that then $\Phi[a,b]_\Phi = (\Phi a) \bullet (\Phi b)$, we see that Φ is hereditary if and only if:

$$\Phi^2(a \bullet b) + (\Phi a) \bullet (\Phi b) = \Phi\{(\Phi a) \bullet b + a \bullet (\Phi b)\} \text{ for all } a, b \in \mathcal{L} \ . \tag{2.5}$$

A linear map $\Phi : \mathcal{L} \to \mathcal{L}$ is said to be **invariant** with respect to $k \in \mathcal{L}$ if

$$0 = k \bullet \Phi(b) - \Phi(b) \bullet k - \Phi(k \bullet b) + \Phi(b \bullet k) \quad \text{for all } b \in \mathcal{L}. \tag{2.6}$$

Any linear map $\Phi : \mathcal{L} \to \mathcal{L}$ which is **left invariant**

$$\Phi(k \bullet b) = k \bullet (\Phi b) \text{ for all } b \in \mathcal{L}. \tag{2.7}$$

with respect to $k \in \mathcal{L}$ as well as **right invariant**

$$\Phi(b \bullet k) = \Phi(b) \bullet k \text{ for all } b \in \mathcal{L}. \tag{2.8}$$

is said to be **super-invariant**. Any Φ which is super-invariant is also invariant.

In order to work with operators Φ, for $k \in \mathcal{L}$ we introduce a map L_k by

$$L_k(\Phi)(b) := k \bullet \Phi(b) - \Phi(b) \bullet k - \Phi(k \bullet b) + \Phi(b \bullet k) \quad \text{for all } b \in \mathcal{L}. \tag{2.9}$$

Hence, Φ is k-invariant if and only if

$$L_k(\Phi) = 0 \tag{2.10}$$

Formally, for $a, b \in \mathcal{L}$ instead of $a \bullet b$ we write

$$L_a(b) := a \bullet b - b \bullet a \tag{2.11}$$

hence

$$L_a(\Phi \bullet b) = L_a(\Phi) \bullet b + \Phi(L_a(b)) \quad \text{for all } b \in \mathcal{L}. \tag{2.12}$$

Using this, one easily finds that L_a is a derivation with respect to operator products, i.e.

$$L_a(\Phi\Psi) = L_a(\Phi)\Psi + \Phi L_a(\Psi). \tag{2.13}$$

However, one should observe that L_a in general is not a derivation on $(\mathcal{L}, \bullet)$. Another important observation is that when Φ is hereditary then

$$\Phi(L_a\Phi) = L_{\Phi(a)}\Phi. \tag{2.14}$$

This is easily seen by direct computation.

Lemma 1 : *Let Φ be hereditary and let it be invariant with respect to k. Then Φ is invariant with respect to $\Phi(k)$. If Φ is invertible then it is invariant with respect to $\Phi^{-1}(k)$ as well. The set $\{k | \Phi$ invariant with respect to $k\}$ of all elements which leave Φ invariant is an invariant subset under the application of Φ (and of Φ^{-1} if Φ is invertible).*

Proof: By invariance of Φ with respect to k we know (see (2.10) and (2.12)) that

$$L_k \Phi a = \Phi L_k a \quad \text{for all } a \in \mathcal{L} \ . \tag{2.15}$$

From this and a twofold application of (2.14) we find for arbitrary $a \in \mathcal{L}$

$$L_{\Phi k}(\Phi a) = \Phi L_k(\Phi a) = \Phi(\Phi L_k a) = \Phi L_{\Phi k} a \tag{2.16}$$

hence

$$L_{\Phi k}(\Phi a) = \Phi L_{\Phi k} a \tag{2.17}$$

which proves the invariance with respect to $\Phi(k)$.

Observation 1 : *Let Φ be hereditary and let it be left invariant with respect to k. Then Φ is left invariant with respect to $\Phi(k)$. If Φ is invertible then it is left invariant with respect to $\Phi^{-1}(k)$ as well. The set $\{k | \Phi$ left invariant with respect to $k\}$ is invariant under the application of Φ (and of Φ^{-1} if Φ is invertible). The same results hold for right invariance.*

Proof: Replace a by k in (2.5). Since Φ is left invariant with respect to k the first and fourth term cancel and the equality reads

$$\Phi(k) \bullet (\Phi b) = \Phi((\Phi k) \bullet b) \text{ for all } b. \tag{2.18}$$

This yields the left invariance with repect to $\Phi(k)$. In case that Φ is invertible, we replace a in (2.5) by $\Phi^{-1}(k)$ and apply Φ^{-1} to the remaining two terms. The proof for right invariance is similar.

Consequence 1 : *Let Φ be hereditary and let it be super invariant with respect to k. Then Φ is super invariant with respect to $\Phi(k)$. If Φ is invertible then it is super invariant with respect to $\Phi^{-1}(k)$ as well. The set $\{k | \Phi$ super invariant with respect to $k\}$ is invariant under the application of Φ (and of Φ^{-1} if Φ is invertible).*

Theorem 1 : *Let Φ be a hereditary map which is invariant with respect to k. Then $\{\Phi^n k | n \in \mathbb{N}_0\}$ is an abelian subset of $(\mathcal{L}, \bullet)$. In case that Φ is invertible then $\{\Phi^n k | n \in \mathbb{Z}\}$ is abelian as well.*

Proof: From lemma 1 we obtain by induction that Φ is invariant with respect to any $\Phi^m k$ and $\Phi^n k$. Consider

$$\begin{aligned}(\Phi^m k) \bullet (\Phi^n k) - (\Phi^n k) \bullet (\Phi^m k) &= L_{\Phi^m k}(\Phi^n k) = \Phi^m L_k(\Phi^n k) \\ &= \Phi^{m+n} L_k(k) \\ &= \Phi^{m+n}(k \bullet k - k \bullet k) \\ &= 0\end{aligned}$$

for all suitable m, n, where (2.14) and the invariance of Φ has been used. This proves that $(\Phi^m k)$ and $(\Phi^n k)$ commute. For invertible Φ, in this argument Φ^{-1} has to replace Φ.

Theorem 2 *Let Φ be a hereditary map which is super-invariant with respect to k_1 and k_2. Then for arbitrary $n, m \in \mathbb{N}$ (or $\in \mathbb{Z}$ if Φ is invertible) we have*

$$\Phi^n(k_1) \bullet \Phi^m(k_2) = \Phi^{n+m}(k_1 \bullet k_2) \tag{2.19}$$

Proof: From consequence 1 we obtain by induction that Φ is super-invariant with respect to any $\Phi^m k_1$ and $\Phi^n k_2$. Hence

$$(\Phi^m k_1) \bullet (\Phi^n k_2) = \Phi^n(\Phi^m(k_1) \bullet k_2) = \Phi^n(\Phi^m(k_1 \bullet k_2)) = \Phi^{n+m}(k_1 \bullet k_2) \tag{2.20}$$

for all suitable m, n. For invertible Φ, in this argument Φ^{-1} has to replace Φ.

Remark 1 : *Let Φ be hereditary and let a_1 and a_2 be eigenvectors of Φ (i.e. $\Phi a_i = \lambda_i a_i$, $\lambda_i \in \mathbf{F}$, $i = 1, 2$). Then for these a_i relation (2.5) is equivalent to*

$$(\Phi - \lambda_1)(\Phi - \lambda_2)(a_1 \bullet a_2) = 0 \ .$$

Hence, in case an operator Φ has a spectral resolution and all the corresponding spectral projections are algebra homomorphisms then this operator is hereditary.

Remark 2 : *One easily sees that Φ is left invariant with respect to k if and only if*

$$[k, b]_\Phi = \Phi(k) \bullet b \quad \text{for all } b \in \mathcal{L} \tag{2.21}$$

and right invariant if and only if

$$[b, k]_\Phi = b \bullet \Phi(k) \quad \text{for all } b \in \mathcal{L} \ . \tag{2.22}$$

Using the definition of hereditaryness we see that a hereditary invertible Φ is super-invariant with respect to k if and only if it k-invariant with respect to $(\mathcal{L}, [\ ,\ ]_\Phi)$.

3 Compatibility

Now, let us return to the general situation of maps from Λ into $\mathcal{L}$, where Λ is a module over $\mathbf{F}$. Assume that in Λ we have $\Theta-$ and $\Psi-$products $[\ ,\ ]_\Theta$ and $[\ ,\ ]_\Psi$, respectively. These products are said to be **compatible** if

$$[a, b] := [a, b]_\Theta + [a, b]_\psi \tag{3.23}$$

defines a $(\Theta + \Psi)$-product.

Lemma 2 : *Let $[\ ,\ ]_\Theta$ and $[\ ,\ ]_\Psi$ be $\Theta-$ and $\Psi-$products, respectively. These products are compatible if and only if*

$$\Theta[a, b]_\Psi + \Psi[a, b]_\Theta = (\Theta a) \bullet (\Psi b) + (\Psi a) \bullet (\Theta b) \ \text{for all } a, b \in \Lambda \ . \tag{3.24}$$

Proof: Observe that $\Theta[a, b]_\Theta = (\Theta a) \bullet (\Theta b)$ and $\Psi[a, b]_\Psi = (\Psi a) \bullet (\Psi b)$. So, (3.24) is obviously equivalent to

$$(\Theta + \Psi)\{[a, b]_\Theta + [a, b]_\Psi\} = ((\Theta + \Psi)a) \bullet ((\Theta + \Psi)b) \ , \tag{3.25}$$

which proves the claim.

Observation 2 : *Let $\lambda \in \mathbf{F}$. Obviously, $[\ ,\ ]_{\lambda\Theta}$ defined by $[a,b]_{\lambda\Theta} := \lambda[a,b]_\Theta$ is a $(\lambda\Theta)$-product whenever $[\ ,\ ]_\Theta$ is a Θ-product. Now, replacing in (3.24) Ψ and $[\ ,\ ]_\Psi$ by $\lambda\Psi$ and $[\ ,\ ]_{\lambda\Psi}$, respectively, we see that (3.24) remains valid. In other words, (3.24) is linear in Ψ (as well as in Θ). Hence, if $[\ ,\ ]_{\Theta_1}$ and $[\ ,\ ]_{\Theta_2}$ are compatible, and if both are compatible with $[\ ,\ ]_\Theta$ then $[\ ,\ ]_{\lambda\Theta_1} + [\ ,\ ]_{\sigma\Theta_2}$ is always compatible with $[\ ,\ ]_\Theta$.*

Observation 3 : *Consider the case that the reference algebra is equal to Λ, i.e. $\Lambda = \mathcal{L}$ and put $\Theta = I, \Psi = \Phi$. Furthermore, assume that $[\ ,\ ]_\Theta$ is the given product in $(\mathcal{L}, \bullet)$, and that $[\ ,\ ]_\Psi = [\ ,\ ]$ is a second product such that $\Phi : (\mathcal{L}, [\ ,\ ]) \to (\mathcal{L}, \bullet)$ is a homomorphism. Then (3.24) holds if and only if $[\ ,\ ]$ is the product defined in (2.3). Hence, Φ is hereditary if and only if $(\mathcal{L}, \bullet)$ and $(\mathcal{L}, [\ ,\ ])$ are compatible.*

In order to shorten our notions we call Ψ and Θ compatible if their Ψ- and Θ-products, $[\ ,\ ]_\Psi$ and $[\ ,\ ]_\Theta$ are compatible. By application of this notion to the special case of hereditary operators Φ_1, Φ_2 we see that Φ_1 and Φ_2 are compatible if and only if $\Phi_1 + \Phi_2$ is again hereditary.

Theorem 3 : *Consider maps $\Theta, \Psi : \Lambda \to \mathcal{L}$ and their corresponding products $[\ ,\ ]_\Theta$ and $[\ ,\ ]_\Psi$. Assume that Ψ is invertible. Then Θ and Ψ are compatible if and only if $\Phi = \Theta\Psi^{-1}$ is hereditary.*

Proof: Define, by use of $[\ ,\ ]_\Psi$ and the invertible Ψ, a second product in $\mathcal{L}$ by

$$[a,b] = \Psi[(\Psi^{-1}a), (\Psi^{-1}b)]_\Theta \text{ for } a, b \in \mathcal{L}$$

Then $\Phi : (\mathcal{L}, [\ ,\ ]) \to (\mathcal{L}, \bullet)$ is a homomorphism. We obtain

$$\begin{aligned}(I+\Phi)(a \bullet b + [a,b]) &= (\Theta + \Psi)\Psi^{-1}((a \bullet b) + [a,b]) \\ &= (\Theta + \Psi)([(\Psi^{-1}a), (\Psi^{-1}b)]_\Psi + [(\Psi^{-1}a), (\Psi^{-1}b)]_\Theta)\ .\end{aligned}$$

For general $a, b \in \mathcal{L}$ this is equal to

$$((\Theta + \Psi)\Psi^{-1}a) \bullet ((\Theta + \Psi)\Psi^{-1}b) = ((I + \Phi)a) \bullet ((I + \Phi)b)$$

if and only if Θ and Ψ are compatible. Hence I and Ψ are compatible if and only if Θ and Ψ are compatible. By use of observation 3 we obtain the required result.

Theorem 4 : *Let Φ, Ψ be compatible hereditary operators and assume that Φ and Ψ do commute. Then $\Phi\Psi$ is hereditary.*

Proof: For completeness we go through the proof although it is almost the same as in [3] (where the situation was more special). Since Ψ and Φ are hereditary we observe (by use of (2.5) and commutativity) that

$$\begin{aligned}(\Psi\Phi a) \bullet (\Psi\Phi b) &= -\Psi^2((\Phi a) \bullet (\Phi b)) + \Psi\{(\Psi\Phi a) \bullet (\Phi b) + (\Phi a) \bullet (\Psi\Phi b)\} \\ &= \Psi^2\Phi^2(a \bullet b) - \Psi^2\Phi\{a \bullet (\Phi b) + (\Phi a) \bullet b)\} \\ &\quad -\Psi\Phi^2\{(\Psi a) \bullet b + a \bullet (\Psi b)\} \\ &\quad +\Psi\Phi\{(\Phi\Psi a) \bullet b + (\Psi a) \bullet (\Phi b) + (\Phi a) \bullet (\Psi b) + a \bullet (\Psi\Phi b)\}\ .\end{aligned}$$

Define a product $[\ ,\ ]_{\Psi\Phi}$ as in (2.3) and insert the last expression into

$$A_{\Phi\Psi}(a,b) = (\Phi\Psi a) \bullet (\Phi\Psi b) - \Phi\Psi[a,b]_{\Psi\Phi}\ .$$

This yields

$$\begin{aligned} A_{\Phi\Psi}(a,b) &= 2(\Phi\Psi)^2(a \bullet b) - (\Phi\Psi)\Psi\{[a,b]_\Phi + \Phi(a \bullet b)\} \\ &\quad -(\Phi\Psi)\Phi\{[a,b]_\Psi + \Psi(a \bullet b)\} \\ &\quad +\Psi\Phi\{(\Psi a) \bullet (\Phi b) + (\Phi a) \bullet (\Psi b)\} \\ &= \Phi\Psi\{-\Psi[a,b]_\Phi - \Phi[a,b]_\Psi + (\Psi a) \bullet (\Phi b) + (\Phi a) \bullet (\Psi b)\} \end{aligned}$$

which vanishes because of the compatibility of Φ and Ψ and by virtue of (3.24). Hence we have $A_{\Phi\Psi}(a,b) = 0$ which gives that $\Phi\Psi : (\mathcal{L}, [\ ,\ ]_{\Phi\Psi}) \to (\mathcal{L}, \bullet))$ must be a homomorphism.

Corollary 1 : *Let Φ be hereditary, then any polynomial in Φ is hereditary.*

Proof: Assume that any polynomial $P(\Phi)$ of degree $\leq N$ in Φ is hereditary (which is certainly true for $N = 1$). Obviously, Φ commutes with $P(\Phi)$ and both are compatible. Thus $\Phi P(\Phi)$ must be hereditary.
From compatibility, with I we conclude that any $\alpha I + \beta\Phi P(\Phi)$ is hereditary. Since any polynomial of degree $(N+1)$ can be written in this form we finish the proof by induction.

Remark 3 : *It should be observed that the notion of compatibility, and therefore the notion of hereditariness as well, are preserved under isomorphisms with respect to their respective reference algebras. To be precise: If $T : (\mathcal{L}, \bullet) \to (\mathcal{L}_1, \bullet)$ is an isomorphism and if $\Theta, \Psi : \Lambda \to \mathcal{L}$ are compatible then $T\Theta, T\Psi : \Lambda \to (\mathcal{L}_1, \bullet)$ are compatible as well.*

Let me add some remarks on nonlinear deformations. Compatibility, as we have defined it, is the tangential structure of a corresponding compatibility notion for continuous deformations of products. Assume that we have a one-parameter family of products $[\ ,\ ]_\lambda$ in Λ and a family of maps $\Theta(\lambda) : \Lambda \to \mathcal{L},$. Assume further that topologies are given such that the occurring quantities are differentiable with respect to λ. We denote

$$\Theta'(\lambda) = \frac{d}{d\lambda}\Theta(\lambda) \tag{3.26}$$

$$[a,b]' = \frac{d}{d\lambda}[a,b]_\lambda \tag{3.27}$$

and we assume that $[\ ,\ ]_\lambda$ is, for any λ, a $\Theta'(\lambda)$ product. We call $(\Theta(\lambda), [\ ,\ ]_\lambda)$ a compatible deformation if $[\ ,\ ]_\lambda$ always is a $\Theta(\lambda)$ product. One easily verifies the following

Remark 4 : *$(\Theta(\lambda), [\ ,\ ]_\lambda)$ is a compatible deformation if and only if $\Theta(\lambda)$ and $\Theta'(\lambda)$ are always compatible.*

4 Antisymmetric Algebras

The algebra $(\mathcal{L}, \bullet)$ is said to be *antisymmetric* if

$$a \bullet b = -b \bullet a \text{ for all } a, b \in \mathcal{L} . \tag{4.28}$$

For any algebra $(\mathcal{L}, \bullet)$ there is a corresponding *antisymmetrization* defined by

$$[\![a, b]\!] := \frac{1}{2}(a \bullet b - b \bullet a) . \tag{4.29}$$

Observe, that $(\mathcal{L}, [\![\ ,\ ]\!])$ not necessarily is a Lie algebra since we did not assume that $(\mathcal{L}, \bullet)$ is associative.

In case $(\mathcal{L}, \bullet)$ itself is already antisymmetric then it is equal to its antisymmetrization. Any homomorphism between two algebras induces a homomorphism between their antisymmetrizations. So any automorphism Φ on $(\mathcal{L}, \bullet)$ canonically defines an automorphism on $(\mathcal{L}, [\![\ ,\ ]\!])$.

Obviously, for antisymmetric algebras the notions *right invariance* and *left invariance* coincide.

Remark 5 *Φ is in $(\mathcal{L}, \bullet)$ super-invariant with respect to k if and only if Φ is in $(\mathcal{L}, [\![\ ,\ ]\!])$ invariant with respect to k.*

Proof: This is easily seen from the following identities

$$[\![k, \Phi a]\!] = L_k(\Phi a) \tag{4.30}$$
$$\Phi[\![k, a]\!] = \Phi L_k(a) \tag{4.31}$$

As a consequence, the notions invariance and super-invariance coincide for antisymmetric algebras.

5 Virasoro Algebras

We show that hereditary operators uniquely correspond to Virasoro algebras. For that we consider some antisymmetric algebra $(\mathcal{L}, [\![\ ,\ ]\!])$. A set $\mathcal{L}_V := \{K_n, \tau_n \mid n \in \mathbb{N}_0\}$ in $\mathcal{L}$ is called the positive part of a Virasoro algebra (or **Virasoro algebra** for short) if there is some $\rho \in \mathbf{F}$ such that for all $n, m \in \mathbb{N}_0$

$$[\![K_n, K_m]\!] = 0 \tag{5.32}$$
$$[\![\tau_n, K_m]\!] = (m + \rho)\, K_{m+n} \tag{5.33}$$
$$[\![\tau_n, \tau_m]\!] = (m - n)\tau_{n+m}. \tag{5.34}$$

Obviously, $\mathcal{L}_V$ is a subalgebra of $(\mathcal{L}, [\![\ ,\ ]\!])$. We consider in $(\mathcal{L}, [\![\ ,\ ]\!])$ the operator Φ defined by

$$\Phi K_n : = K_{n+1} \tag{5.35}$$
$$\Phi \tau_n : = \tau_{n+1} \tag{5.36}$$
$$\tag{5.37}$$

Theorem 5 Φ *is hereditary.*

Proof: We consider the algebra $[\ ,\ ]_\Phi$ as defined in (2.4)

$$[a,b]_\Phi := (\Phi a)\bullet b + a\bullet(\Phi b) - \Phi(a\bullet b)\ . \tag{5.38}$$

Then we find

$$\begin{aligned}
\Phi[K_n,K_m]_\Phi : &= \Phi\{[\![K_{n+1},K_m]\!] + [\![K_n,K_{m+1}]\!] - \Phi[\![K_n,K_m]\!]\} \\
&= 0 \\
\Phi[\tau_n,K_m]_\Phi : &= \Phi\{[\![\tau_{n+1},K_m]\!] + [\![\tau_n,K_{m+1}]\!] - \Phi[\![\tau_n,K_m]\!]\} \\
&= (m+1+\rho)\Phi K_{n+m+1} = [\![\tau_{n+1},K_{m+1}]\!] \\
&= [\![\Phi\tau_n,\Phi K_m]\!] \\
\Phi[\tau_n,\tau_m]_\Phi : &= \Phi\{[\![\tau_{n+1},\tau_m]\!] + [\![\tau_n,\tau_{m+1}]\!] - \Phi[\![\tau_n,\tau_m]\!]\} \\
&= (m-n)\Phi\tau_{n+m+1} = [\![\tau_{n+1},\tau_{m+1}]\!] \\
&= [\![\Phi\tau_n,\Phi\tau_m]\!]
\end{aligned}$$

In order to show that hereditary operators also lead to Virasoro algebras we start with $\tau_0, K_0 \in \mathcal{L}$ and some Φ such that for some $\rho \in \mathbf{F}$

$$L_{\tau_0}\Phi = \Phi, \quad L_{\tau_0}K_0 = \rho K_0 \tag{5.39}$$

Then we define

$$K_n : = \Phi^n K_0 \tag{5.40}$$

$$\tau_n : = \Phi^n \tau_0 \tag{5.41}$$

(5.42)

Theorem 6 $\mathcal{L}_V := \{K_n, \tau_n \mid n \in \mathbb{N}_0\}$ *is a Virasoro algebra.* $L_{\tau_n}\Phi$ *is hereditary.*

Proof: The Virasoro algebra property is easily proved by induction with the use of (2.14). The same relation shows then

$$L_{\tau_n}\Phi = \Phi^n \tag{5.43}$$

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