Physica, D 95, 1996 p.229-243

Some tricks from the Symmetry-Toolbox for Nonlinear equations:

Generalizations of the Camassa-Holm equation

Benno Fuchssteiner
University of Paderborn, D 4790 Paderborn, Germany [1]

Abstract: The main subject of the paper is to give a survey and to present new methods on how integrability results (i.e. results for symmetry groups, inverse scattering formulations, action-angle transformations and the like) can be transferred from one equation to others in case the equations are NOT related by Bäcklund transformations. As a main example the so called Camassa-Holm equation is chosen for which the relevant results are obtained by having a look on the KdV. The Camassa-Holm equation turns out to be a *different-factorization equation* of the KdV, it describes shallow water waves and reconciles the properties which were known for different orders of shallow water wave approximations. We follow here an old method already marginally mentioned in [9] and [10] and recently applied by others [23]. The method allows an immediate recovery of the recursion operator for the Camassa-Holm equation from the invariance structure of the KdV, although both equations are not related by Bäcklund transformations. However, in addition, and different from other approaches, from there by use of the squared eigenfunction relation for the Korteweg de Vries equation, the Lax pair formulation for the different-factorization-equation is derived. For the example under consideration it is, of course, the one obtained in [5]. Since the methods proposed here can be transferred to any compatible factorization of recursion operators its application, even in the special case which was chosen for illustration, leads to a large class of integrable equations among which the Camassa-Holm equation can be found as well as a three-parameter family of generalizations of that equation (see section 3). The advantage of the general approach to the Lax pair presentation is that direct transformations between action and angle variables are obtained. So, using this Lax pair formulation, as a novel result, a direct transformation between action- and angle-variables for the Camassa-Holm equation is derived. Further novel results in the paper are: a hodograph link back from a Bäcklund transformation of the Camassa-Holm equation to a particular member of the KdV-hierarchy, additional symmetries, and the construction of the conformal algebra for the hierarchy of the Camassa-Holm equation. The methods involved include: Hereditary operators, bi-Hamiltonian formulations, nilpotent flows and scaling symmetries.

[1] Major parts of this paper were written while visiting the University of Colorado at Boulder

1 Introduction

Approximations of the Euler equation for an incompressible fluid of uniform density and vanishing viscosity have lead to many interesting nonlinear systems (compare [??],[??]). The fascination of these equations for the mathematician is that, among other features, these equations differ dramatically in their structural properties. Among these are the shallow water approximations of lowest order, which lead to a two-component system, also known from nonlinear gas dynamics,

$$\begin{aligned} (hv)_x + h_t &= 0 \\ \frac{1}{2}(v^2)_x + h_x + v_t &= 0 \; . \end{aligned}$$

From the symmetry point of view these equations are not overly interesting, however they admit discontinuities which break energy conservation thus for their discontinuous solutions they are not hamiltonian whereas on a manifold of continuous functions the system is hamiltonian[12], a phenomenon well worth to study in case of infinitely many symmetries.

The approximation of first order leads to the well known Korteweg de Vries equation

$$u_t = u_{xxx} + 6u_x u \; .$$

This first order equation has infinitely many symmetries of polynomial type, however a slight variation of the second order approximation (as given by the Benjamin-Bona-Mahony equation [1]), although it admits soliton-like solutions, cannot have an infinite number of symmetries because there is no complete energy conservation for the emerging solitons [2]. So it is an intriguing idea to assume that when going to even higher order approximations, then again all these properties, like solitons and infinitely many symmetries may be found.

Indeed, such an equation, for which the author of the present paper was looking for some time, has been found by Camassa and Holm [5]. This equation, or rather a suitable generalization thereof, has the potential to become the master equation for shallow water wave theory in one spatial dimension. In particular, because most of the relevant structure one expects from an integrable system has been shown for it. In addition this equation may eventually allow to study the breaking of conservation laws by discontinuities in case of infinite-dimensional abelian symmetry groups.

In their paper [5] the authors start from the Euler equation, with free surface and under the influence of gravity, and derive from there a two-component system (already known in the literature [21]). Then they make a further unidirectional wave approximation in order to reduce that equation to a one-component system. The resulting system looks very much like the Benjamin-Bona-Mahony equation, only that some terms are added, which would be absent if approximations of the order of the KdV or the BBM are considered. However, as shown in [5],the importance of these additional terms is that they ensure the complete integrability of the resulting system; furthermore these terms are responsible for the admissibility of peakons, i.e. limiting solutions with sharp peaks at which there are discontinuous derivatives.

In this paper we obtain the integrability of this equation by using known results for the Korteweg de Vries equation, see also [23]; this may be surprising insofar as no Bäcklund transformation between the KdV and this new equation exists. The equation turns out to be the *different-factorization equation* of the KdV. However, in addition to known or old results we present additional construction techniques for further invariants of different-factorization equations. The principal tool for showing its integrability is a compatible factorization of the KdV-recursion operator. The structural results known for hereditary operators, then allow an immediate derivation of the recursion operator for the Camassa-Holm equation. From there, using the squared eigenfunction relation for the Korteweg de Vries equation, the Lax pair formulation for the new equation is obtained. The method for arriving at the Lax pair, although only carried out for for the Camassa-Holm equation is general and can be applied to other equations as well. Then, using the Lax pair formulation, a direct transformation between canonical action and angle variables for the new equation is derived; furthermore a hodograph link back to a particular member of the KdV hierarchy is exhibited; further symmetries are discovered the construction of the positive-index part of a Virasoro algebra for the hierarchy is carried out.

Ironically, when looking at the methods of the present paper and comparing them with the remarks appended in [10], one will find out that the new hierarchy already could have been given therein, but alas, by carelessness the coefficients of that equation were not computed correctly, so the equation found there (which even was termed *generalized BBM*) could not serve the purpose of reconciling the different properties which were known for different approximations of shallow water wave theory.

2 The main example: Integrability of the Camassa-Holm equation

2.1 Background

In the following we present some background material which is needed to understand the constructions carried out subsequently. We restrict this introduction to the basics, since some readers are familiar with that material anyway. For a more detailed account we refer to the survey [15].

We consider flows on manifolds. Instead of working with directional derivatives

$$F'[v] \;=\; F'(u)[v] := \frac{\partial}{\partial\epsilon}_{|\epsilon=0} F(u \;+\; \epsilon v). \tag{2.1}$$

of quantities F at the manifold-point u in the direction of v, we prefer, for good reason, a differential geometric invariant formulation. Such an invariant formulation for vector fields is given by considering instead the Lie algebra with respect to commutators. This is defined (for arbitrary vector fields K, G) by

$$
\begin{aligned}
[K,G](u) : \;&= \frac{\partial}{\partial \epsilon}_{|\epsilon=0} \{G(u \;+\; \epsilon K(u)) \;-\; K(u \;+\; \epsilon G(u))\} \\
&= G'(u)[K(u)] \;-\; K'(u)[G(u)]. \qquad (2.2)
\end{aligned}
$$

Carrying over this structure to the corresponding Lie module, and to higher order tensors yields a coordinate-independent formulation for the tangential structure of the manifold under consideration. For that we need the notion of *Lie derivative*. For a vector field K this derivative in direction G is defined by the commutator $[G, K]$. For scalar fields f, the Lie derivative is given by taking the directional derivative $f'[G]$, and for all other tensors we require the validity of the *product rule*. This completely defines the Lie derivatives for all tensors (i.e. for all multilinear forms on $(\otimes\mathcal{L}^*)^r \otimes (\otimes\mathcal{L})^n,\ n, r\mathbb{N}$ where $\mathcal{L}$ and $\mathcal{L}^*$ denote the vector fields and the covector fields, respectively. Such multilinear forms are called n-times *covariant* and r-times *contravariant tensors*).

To illustrate the construction of the Lie derivative L_K, we compute Lie derivatives for 1-times covariant tensors and for 2-times contravariant tensors, i.e. for covector fields and for linear operators from the cotangent bundle to the tangent bundle. To do that, we first compute the Lie derivative for some fixed $\gamma \in \mathcal{L}^*$ (which may be considered as a linear map on $\mathcal{L}$ or a 1-times covariant tensor):

So, denote by $<,>$ the canonical scalar product between $\mathcal{L}$ and $\mathcal{L}^*$ and consider $\gamma \in \mathcal{L}^*$ and $G \in \mathcal{L}$. The product rule, applied to $< \gamma, G >$, gives

$$
< L_K(\gamma), G >= L_K < \gamma, G > - < \gamma, [K, G] > \ .
$$

Therefore, the linear map $L_K(\gamma) : \mathcal{L} \to \mathcal{F}$ is defined by

$$
L_K(\gamma) = L_K \cdot \gamma - \gamma \cdot L_K \ . \qquad (2.3)
$$

As another example we take some linear operator $\Theta : \mathcal{L}^* \to \mathcal{L}$. Its Lie derivative L_K we compute again by the product rule applied to $< \gamma_1, \Theta\gamma_2 >$ where γ_1, γ_2 are arbitrarily chosen elements in $\mathcal{L}^*$. This yields

$$
< \gamma_1, L_K(\Theta)\gamma_2 >= L_K < \gamma_1, \Theta\gamma_2 > - < L_K(\gamma_1), \Theta\gamma_2 > - < \gamma_1, \Theta L_K(\gamma_2) > \qquad (2.4)
$$

On the right side, the Lie-derivative of the first term is given by its definition on scalar fields, and the Lie derivatives of the γ's were determined by (2.3). Since γ_1 and γ_2 were arbitrary, (2.4) defines completely the Lie derivatives for the two-times contravariant tensor Θ. In the same way we can define, by induction, the derivative L_K for arbitrary tensors. In this way the Lie derivative may be computed on all tensors.

A tensor T is said to be *invariant with respect to the vector field* K if $L_K T = 0$. One-time-covariant-one-time-contravariant tensors being invariant with respect to a vector field K are called *recursion operators* for $u_t = K(u)$. In many cases the construction of recursion operators is carried out with the help of hereditary operators.

A linear $\Phi : \mathcal{L} \to \mathcal{L}$ is said to be *hereditary* if

$$\Phi^2[K,G] + [\Phi K, \Phi G] = \Phi\{[\Phi K, G] + [K, \Phi G]\} \ for\ all\ K, G \in \mathcal{L} \tag{2.5}$$

or equivalently, if for arbitrary vector fields K the corresponding Lie derivative fulfills

$$\Phi L_K \Phi = L_{\Phi K} \Phi \ . \tag{2.6}$$

Well known is the following result:

Theorem: 1 *Let Φ be a hereditary map which is invariant with respect to K. Then $\{\Phi^n K | n \in \mathbb{N}_0\}$ is an abelian subset of $(\mathcal{L},\ [\ ])$. If Φ is invertible then $\{\Phi^n K | n \in \mathbb{Z}\}$ is abelian as well.*

Thus hereditary operators generate hierarchies of commuting flows; examples for application of this construction are most hierarchies of integrable flows in 1+1-dimension.

There are ways to compute, from known hereditary operators, others. For example, if Φ is hereditary then, if existent, also Φ^{-1} is hereditary. Another way to construct new hereditary operators, is given by the notion of compatibility. Two hereditary operators Φ and Ψ are said to be *compatible* if $\Psi + \Phi$ again is hereditary. It is easy to see that when Φ and Ψ are compatible, then also Φ^{-1} and Ψ^{-1} are. The following is known [15] and will be used extensively in the sequel:

Observation: 1 *Let Φ, Ψ be compatible hereditary operators then $\Phi\Psi^{-1}$ and $\Psi^{-1}\Phi$ are hereditary.*

2.2 Application

We prove the integrability of the Camassa-Holm equation by a simple and transparent method. We consider the Korteweg de Vries equation

$$u_t = u_{xxx} + 6uu_x$$

which may be written as

$$u_t = \Phi(u)u_x$$

where $\Phi(u)$ is the hereditary operator

$$\Phi(u) = D^2 + 2DuD^{-1} + 2u$$

which, by the hereditary property together with translation invariance, must be the recursion operator of the Korteweg de Vries equation.

Now, using the trivial Bäcklund transformation

$$u \to u + c, \quad c \text{ an arbitrary constant}$$

and rescaling u and t, we obtain the equation

$$u_t = \alpha u_{xxx} + 6\beta uu_x + \gamma u_x \tag{2.7}$$

which again may be written, by use of a suitable hereditary operator, as

$$u_t = \Phi_{(\alpha,\beta,\gamma)}(u)u_x$$

where

$$\Phi_{(\alpha,\beta,\gamma)}(u) = \alpha D^2 + 2\beta D u D^{-1} + 2\beta u + \gamma \ . \tag{2.8}$$

Hence, $\Phi_{(\alpha,\beta,\gamma)}(u)$ is the recursion operator of (2.7).

Specializing the constants we find that

$$\hat{\Phi} = \Phi_1(u) + c\Phi_2$$

where

$$\Phi_1 = DuD^{-1} + u + 2k, \quad \text{and } \Phi_2 = (D^2 - I)$$

must be, for all c, hereditary. Hence, as seen in the last subsection,

$$\Psi = \Phi_1(u)\Phi_2^{-1} \tag{2.9}$$

is hereditary. Therefore, due to translation invariance, the equation

$$u_t = \Psi(u)u_x \tag{2.10}$$

is integrable. We call it the *factorized KdV*, its recursion operator is $\Psi(u)$ and an abelian sequence of symmetries is given by

$$K_n := (\Psi(u))^n u_x \ .$$

Now, we take the Bäcklund transformation

$$u := (v - v_{xx}) = (I - D^2)v \tag{2.11}$$

to go over from a manifold with variable u to a manifold where the general variable is denoted by v. Thus equation (2.10) reads in the coordinates given by the new variable:

$$v_t - v_{xxt} = -\Phi_1(v - v_{xx})v_x \ .$$

Performing the differentiations we obtain explicitly

$$v_t - v_{xxt} = -D[(v - v_{xx})v] - (v - v_{xx})v_x$$

or

$$v_t - v_{xxt} = -3vv_x + 2v_{xx}v_x + v_{xxx}v. \tag{2.12}$$

By the trivial shift $v \to v - c_0$ becomes

$$v_t - v_{xxt} = 3c_0 v_x - 3vv_x + 2v_{xx}v_x + v_{xxx}v - c_0 v_{xxx} \tag{2.13}$$

which is the Camassa-Holm equation equation [5] and [4]. Since, by manifold transformations, invariances and properties like hereditariness are preserved, the hereditary recursion operator for that equation is easily obtained by the transformation formulas for tensors being one-time-covariant and one-time-contravariant (see also [8] for the explicit transformation of such quantities). Transforming the corresponding operator $\Psi(u)$ of (2.10) we obtain the recursion operator $\Psi_{CH}(v)$ for the Camassa-Holm equation as

$$\begin{aligned}\Psi_{CH}(v) &= (I-D^2)^{-1}\Psi(u)(I-D^2)\\ &= -(I-D^2)^{-1}\Phi_1(v-v_{xx})\\ &= (D^2-I)^{-1}\{D(v-v_{xx})D^{-1}+(v-v_{xx})\}\ . \end{aligned} \tag{2.14}$$

Thus the hierarchy of the Camassa-Holm equation is given by the vector fields

$$K_n(v) := (\Psi_{CH}(v))^n v_x\ . \tag{2.15}$$

3 A new class of equations of similar kind

The same process can be performed with other equations, for example with modifications of the KdV. Take the Gardner equation (that is a linear combination of KdV and mKdV) together with its hereditary recursion operator

$$\begin{aligned} u_t &= \Phi_{Gardner}(u)u_x\\ &= u_{xxx}+6uu_x+6c\ u^2u_x \end{aligned} \tag{3.1}$$

where

$$\Phi_{Gardner}(u) = D^2+2DuD^{-1}+2u+4c\ DuD^{-1}u\ . \tag{3.2}$$

Again performing elementary substitutions, we obtain a hereditary operator with free constants

$$\Phi_{Gardner\_scaled}(u) = \alpha D^2+\beta(2DuD^{-1}+2u)+\gamma DuD^{-1}u+\delta\ . \tag{3.3}$$

Specializing these constants we find that the following sum always is hereditary

$$\Phi_{Gardner\_sum}(u) := \Phi_1(u)+c\ \Phi_2(u),\ \ c \text{ arbitary}$$

where

$$\begin{aligned}\Phi_1(u) &= \alpha D^2+\beta(DuD^{-1}+u)+2\gamma DuD^{-1}u+2k\\ \Phi_2(u) &= (I-D^2)\end{aligned}$$

By using the same factorization arguments as before we find that

$$\Phi_{Gardner\_factor}(u) := \Phi_1(u)\Phi_2(u)^{-1}\ . \tag{3.4}$$

must be hereditary. Now taking the same manifold transformation as before we obtain in generalization of the Camassa-Holm equation an integrable hierarchy. The first member of that hierarchy of generalized Camassa-Holm equation is

$$\begin{aligned} v_t - v_{xxt} &= \Phi_1(v-v_{xx})v_x \qquad (3.5)\\ &= \alpha v_{xxx}-\beta(3v_xv-2v_xv_{xx}+v_{xxx}v)+\gamma\{(v-v_{xx})(v^2-v_x^2)\}_x \end{aligned}$$

or in a more explicit form

$$\begin{aligned} v_t - v_{xxt} &= \alpha v_{xxx} - \beta(3v_x v - 2v_x v_{xx} + v_{xxx} v) \\ &\quad +\gamma(3v_x v^2 - 4v v_x v_{xx} - v_{xxx} v^2 - v_x^3 + v_{xxx} v_x^2 + 2v_{xx}^2 v_x) \end{aligned} \tag{3.6}$$

The recursion operator for that equation is

$$\Psi_{modCH}(v) = (D^2 - I)^{-1}\Phi_1(v - v_{xx}) \tag{3.7}$$

where the nonlocal operator $(I - D^2)^{-1}$, by working with fundamental solutions of differential equations, is easily represented [5] as

$$\left((I - D^2)^{-1}\phi\right)(x) = \frac{1}{2}\int_{-\infty}^{+\infty} \exp(-|x-y|)\phi(y)dy \tag{3.8}$$

4 Bi-hamiltonian structure

We recall that an operator-valued function $\Theta(u)$ mapping each manifold element u to some linear operator $\Theta(u)\ : T_uM^* \to T_uM$ from the cotangent to the tangent bundle is called *implectic operator* if the bracket among scalar fields F_1, F_2 defined by

$$\{F_1, F_2\}_\Theta := < \nabla F_2, \Theta^\circ \nabla F_1 > \tag{4.1}$$

is a Lie-algebra. Here ∇ denotes the gradient defined by

$$< \nabla F, K >= L_K F\ .$$

These brackets are called *Poisson brackets*. The name implectic derives from the fact that algebraically these operators behave like the inverse of symplectic operators, i.e. like the inverse of a closed two-form. If an operator is implectic then $\Theta^\circ\nabla$ defines a Lie algebra homomorphism from the Poisson brackets into the vector field Lie algebra, i.e. we have

$$\Theta^\circ \nabla \{F_1, F_2\}_\Theta \ = \ [\Theta^\circ \nabla F_1, \Theta^\circ \nabla F_2]\ . \tag{4.2}$$

An equation

$$u_t = \Theta(u)\Gamma(u)$$

is said to be *hamiltonian* if $\Gamma(u)$ is a closed covector field and if Θ is implectic. Two implectic operators are said to be *compatible* if their sum is again implectic (see [18], [19], [20] or [9], [10] or the survey [15]). In the survey [15, theorem 4.5 and 6.11] one also will find proofs of generalizations of the following remarks:

Remark: 1 *Let Θ be invertible and implectic, and consider the vector field $K = \Theta\Gamma$ (where Γ is a covector field). Then Θ is invariant with respect to K, i.e. $L_K\Theta = 0$, if and only if Γ is closed.*

Remark: 2 *Let Θ_1 and Θ_2 be implectic.*

1. *If $\Theta_1 + c\Theta_2$ is again implectic, i.e. the operators are compatible, and if Θ_1 is invertible, then*
$$\hat{\Theta} = \Theta_2 \Theta_1^{-1} \Theta_2$$
is implectic.

2. *The sum $\Theta_1 + c\Theta_2$ is again implectic if and only if $\Theta_2\Theta_1^{-1}$ is hereditary. The factorization given for this hereditary operator then is called a* symplectic-implectic factorization.

First, we concentrate on the bi-hamiltonian structure of the Camassa-Holm equation, later on a transfer to the equation (3.6) is performed. only (since a transfer of the corresponding results to the equation (3.6) is trivial and obvious). It is well known that for the KdV the implectic operators

$$\begin{aligned}\Theta_{KdV\_1}(u) &= D^3 + 2Du + 2uD \\ \Theta_{KdV\_2}(u) &= D\end{aligned}$$

provide hamiltonian formulations, and that these two operators are compatible. Hence their sum is implectic, and a rescaling of the operator thus obtained implies that
$$\Theta_1(u) = \alpha D^3 + 2\beta(Du + uD) + \gamma D$$
is implectic. Specializing the free constants we find that

$$\Theta(u) = \Theta_1(u) + c\Theta_2(u)$$

where
$$\Theta_1(u) = (Du + uD + kD), \quad \Theta_2(u) = (D^3 - D) \tag{4.3}$$
is a compatible sum of implectic operators (these are the two compatible hamiltonian operators found in [5]). Hence the hereditary operator in equation (2.9) has the following symplectic-implectic factorization

$$\Psi(u) = (Du + uD + kD)(D^3 - D)^{-1} \tag{4.4}$$

Since the factors in (4.4) are translation invariant, both $\Theta_1(u)$ and $\Theta_2(u)$ are invariant with respect to the generator of translation $K(u) = u_x$. By the same argument the operator $\Theta_2\Theta_1^{-1}\Theta_2$ is invariant with respect to u_x. Hence the covector fields

$$\begin{aligned}\Gamma_1(u) &= \Theta_2^{-1} u_x \\ \Gamma_2(u) &= \Theta_2^{-1}\Theta_1\Theta_2^{-1} u_x\end{aligned} \tag{4.5}$$

are closed (Remark 1). So these fields must have potentials. These are easily computed
$$P_i(u) = \int_0^1 \int_{\mathbb{R}} \Gamma_i(\lambda u) u \; dx \; d\lambda \; . \tag{4.6}$$

As a consequence equation (2.10)

$$u_t = \Theta_1(u)\Theta_2^{-1}(u)u_x = \Psi(u)u_x$$

has the bi-hamiltonian formulation

$$u_t = \Theta_2(u)\nabla P_2(u) = \Theta_1(u)\nabla P_1(u)\ . \tag{4.7}$$

Since manifold transformations do not change the bi-hamiltonian nature of an equation, we find the following bi-hamiltonian formulations of the Camassa-Holm equation

$$v_t = \Theta_{CH\_1}(v)\nabla P_1(v - v_{xx}) \tag{4.8}$$

$$v_t = \Theta_{CH\_2}(v)\nabla P_2(v - v_{xx}) \tag{4.9}$$

where

$$\Theta_{CH\_i}(v) = (I - D^2)^{-1}\Theta_i(v - v_{xx})(I - D^2)^{-1}$$

are the compatible implectic operators obtained from (4.3) by use of the manifold transformation under consideration. Carrying out the integration in (4.6) we obtain for the $P_i's$:

$$P_1(v - v_{xx}) = \frac{1}{2}\int_{\mathbb{R}} (v_{xx} - v)v dx \tag{4.10}$$

$$P_2(v - v_{xx}) = \frac{1}{2}\int_{\mathbb{R}} (v^3 + v v_x^2) dx\ . \tag{4.11}$$

These can be found in [5].

Now, for the bi-hamiltonian formulation of the generalized equations we proceed similarly: We observe that for the rescaled Gardner equation the implectic operator

$$\Theta_{Gardner\_scaled}(u) = \alpha D^3 + \beta(2Du + 2uD) + \gamma D u D^{-1} u D + \delta D$$

provides a hamiltonian formulation. Specializing the free constants we find that

$$\Theta_{GF}(u) = \Theta_{GF\_1}(u) + c\Theta_{GF\_2}(u)$$

where

$$\Theta_{GF\_1}(u) = \alpha D^3 + \beta(Du + uD) + 2\gamma D u D^{-1} u D + 2kD \tag{4.12}$$

$$\Theta_{GF\_1}(u) = (D - D^3) \tag{4.13}$$

is a compatible pair of implectic operators which obviously provides a symplectic-implectic factorization of the hereditary operator $\Phi_{Gardner\_factor}(u)$

$$\Phi_{Gardner\_factor}(u) = \Theta_{GF\_1}\Theta_{GF\_2}^{-1} \tag{4.14}$$

By the same arguments as before we find the bi-hamiltonian formulation of

$$u_t = \Phi_{Gardner\_factor}(u)u_x$$

as

$$u_t = \Theta_{GF\_2}(u)\nabla P_{GF\_2}(u) = \Theta_{GF\_1}(u)\nabla P_{GF\_1}(u) \ . \tag{4.15}$$

where the conservation laws

$$P_{GF\_i}(u) = \int_0^1 \int_{\mathbb{R}} \Gamma_{GF\_i}(\lambda u)u \ dx \ d\lambda \tag{4.16}$$

are the potentials of

$$\begin{aligned} \Gamma_{GF\_1}(u) &= \Theta_{GF\_2}^{-1} u_x \\ \Gamma_{GF\_2}(u) &= \Theta_{GF\_2}^{-1}\Theta_{GF\_1}\Theta_{GF\_2}^{-1} u_x \end{aligned} \tag{4.17}$$

Now, using the manifold transformation $u := (v - v_{xx}) = (I - D^2)v$ we find the following bi-hamiltonian formulation of the generalized equation (3.6)

$$v_t = \Theta_{GCH\_1}(v)\nabla P_1(v - v_{xx}) \tag{4.18}$$

$$v_t = \Theta_{GCH\_2}(v)\nabla P_2(v - v_{xx}) \tag{4.19}$$

where

$$\Theta_{GCH\_i}(v) = (I - D^2)^{-1}\Theta GF\_i(v - v_{xx})(I - D^2)^{-1}$$

are the compatible implectic operators obtained from (4.12) -(4.13) by use of the manifold transformation under consideration. Carrying out the integration in (4.16) we obtain for the $P_{GF\_i}{}'s$:

$$P_{GCH\_1}(v - v_{xx}) = \frac{1}{2}\int_{\mathbb{R}} (v_{xx} - v)v dx \tag{4.20}$$

$$\begin{aligned} P_{GCH\_2}(v - v_{xx}) = \int_{\mathbb{R}} \Big[&\frac{\alpha}{2} v v_{xx} + \frac{\beta}{6}(3v^2 - {v_x}^2 - 2v v_{xx}) \\ &+ \frac{\gamma}{4}(v^3 - v_{xx}v^2 - v{v_x}^2 + v_{xx}{v_x}^2) + k v_x \Big] dx \ . \end{aligned} \tag{4.21}$$

5 Isospectral formulation

Observe that for any equation

$$v_t = G(v)$$

which has a recursion operator, i.e. for which there is some covariant-contravariant tensor $\Psi(v)$ with $L_G\Psi = 0$, there is a Lax pair formulation automatically given:

$$\Psi(v)_t = [G'(v), \Psi(v)] \ . \tag{5.1}$$

Hence recursion operators provide isospectral formulations. Sometimes, however this isospectral formulation is cumbersome, so it is desirable to change it into a lower order problem. We shall carry out this reduction for the Camassa-Holm equation.

Consider an eigenvector φ of Ψ_{CH} with eigenvalue λ

$$\Psi_{CH}(v)\ \varphi = \lambda\varphi\ .$$

Using (2.14) we can rewrite that as

$$(I - D^2)^{-1}\Psi(u)(I - D^2)\varphi = \lambda\varphi$$

where $u = v - v_{xx}$. Introducing

$$\phi := (I - D^2)\varphi$$

this is rewritten as

$$\Psi(u)\phi = \lambda\phi$$

or, by using (2.9), as

$$\Phi_1(u)(I - D^2)^{-1}\phi = -\lambda\phi\ .$$

Introducing φ again, we have

$$(\Phi_1(u) + \lambda(I - D^2))\varphi = 0$$

or

$$(D^2 - \frac{1}{\lambda}\Phi_1(u))\varphi = \varphi \tag{5.2}$$

or

$$(D^2 + 2D\tilde{u}D^{-1} + 2\tilde{u})\varphi = \varphi \tag{5.3}$$

where

$$\tilde{u} = -\frac{u}{2\lambda}\ .$$

Observe that the left-hand side of (5.3) is the recursion operator for the Korteweg de Vries equation (only u replaced by $-u/(2\lambda)$). For this recursion operator there is a well known reduction to a lower order problem.

Remark: 3 *Consider $L(u) = D^2 + u$, then ω is an eigenvector of L with eigenvalue λ if and only if $(\omega^2)_x$ is an eigenvector of $\Phi(u) = D^2 + 2DuD^{-1} + 2u$ with eigenvalue 4λ.*

Application of this remark, which can be verified by a simple direct computation, yields

Observation: 2 *Consider*

$$L_1(u) = D^2 - \frac{u}{2\lambda}$$

then ω is an eigenvector of $L_1(u)$ with eigenvalue $1/4$ if and only if

$$\varphi = (\omega^2)_x$$

solves the eigenvector problem (5.2).

Consequence: 1 *The following are equivalent:*

1) ω *solves*

$$\left(D^2 - \frac{v - v_{xx}}{2\lambda} - \frac{1}{4}\right)\omega = 0 \tag{5.4}$$

2) $\varphi = (\omega^2)_x$ *is an eigenvector of the recursion operator of the Camassa-Holm equation equation with eigenvalue* λ.

3) $\phi = (I - D^2)\varphi$ *is an eigenvector of the recursion operator of the factorized KdV equation with eigenvalue* λ.

As a result we see that the eigenvalue problem (5.4) is isospectral under the flow of the Camassa-Holm equation and that the relation $\varphi = (\omega^2)_x$ is the squared eigenfunction relation for that equation.

For the reader it is left as an exercise to compute the second operator $B(u)$ such that for

$$\Lambda(u) = D^2 - \frac{v - v_{xx}}{2\lambda} - \frac{1}{4}$$

we have

$$\Lambda(v)_t = B(u)\Lambda(u) - \Lambda(u)B(u)\ .$$

In principle, this is simple, one has to take the dynamics of the eigenfunctions of the recursion operator, and to transform that to the dynamics of the eigenvector of (5.4). Recall that the dynamics of the eigenvectors of the recursion operator is given by the variational derivative of the vector field for the flow under consideration (see [7]). Both, the isospectral problem (5.4) as well as the associated evolution equation for the eigenvector (compatibility condition) are given in [5], by completely different methods however. The method presented here may be applied to all equations generated in this way. Observe that application of the trivial shift $v \to v - c_0$, which corresponds to a change in boundary conditions, gives also the isospectral problem for (2.13).

6 Action-angle variables and invariances of the spectral problem

From the theory of recursion operators we know that the eigenvectors can be understood as infinitesimal generators of one-parameter symmetry groups [7]. Transferring these, via the inverse of the implectic operator given by one of the hamiltonian formulations, to the covector bundle we obtain covector fields which are closed because they are the gradients of the corresponding spectral value. So take the eigenvector φ of

$$\Psi_{CH}(v)\ \varphi = \lambda\varphi$$

then

$$s = (\Theta_{CH\_2}(v))^{-1}\varphi$$

can be understood as the gradient of an action variable. One should observe that this means that whenever ω fulfills

$$\left(D^2 - \frac{v - v_{xx}}{2} - \frac{1}{4}\right)\omega = 0 \tag{6.1}$$

with vanishing boundary conditions at infinity then

$$s = (I - D^2)^{-1}\omega^2$$

is a closed covector field which is invariant under the flow of the Camassa-Holm equation, i.e. it is the gradient of an action variable. Furthermore, it is a Casimir field with respect to a linear combination of the corresponding implectic operators, i.e. it fulfills the *annihilation equation*

$$(\Theta_{CH\_1} - \lambda\Theta_{CH\_2})s = 0 \ . \tag{6.2}$$

All action variables fulfilling this do commute with respect to the Poisson brackets (consequence of the hereditariness of the operator having the $\Theta_{CH\_1,2}$ as symplectic-implectic factorization). Hence, these eigenvectors are canonical candidates for an action-angle representation of the system, only the corresponding angle variables are missing. The aim of the subsequent considerations is to find these missing quantities.

We claim that there is a *fundamental equation*, which is an additional invariance for the system, and which determines the corresponding angle variables automatically. This fact is a consequence of lengthy considerations which are presented below. In order to show that these lengthy arguments really lead to new facts we first present, as a recipe, the method resulting from them.

Method:
Consider a new independent variable y and the following partial differential equation

$$(\sigma - \sigma_{xx})_{yx}(\sigma - \sigma_{xx}) - (\sigma - \sigma_{xx})_y(\sigma - \sigma_{xx})_x = (\sigma - \sigma_{xx}) \tag{6.3}$$

The importance and the origin of this equation is shown below (see (6.16)).

We now proceed in the following way:

We determine the eigenvector $s(x)$ of (6.2), then we prescribe this function as initial condition $\sigma(x, y = 0) = s(x)$ for equation (6.3). Solving the initial value problem (which is easy) produces a function $\sigma(x, y)$. For arbitrary constants $y = y_0$ we claim that the function

$$\tilde{s}(x) := \sigma(x, y = y_0)$$

is again a solution of (6.2). In particular that

$$\hat{s} := \sigma_y(x, y = 0)$$

is the gradient of the angle variable correponding to s.

It should be mentioned that similar results hold for most integrable systems. For example, to obtain the corresponding result for the Korteweg de Vries equation [14] one only has to replace (6.3) by

$$\sigma_{yx}\sigma - \sigma_y\sigma_x = \sigma \ . \tag{6.4}$$

Justification of the method: We treat the general case, where the sum

$$\Theta := \Theta_1 - \lambda\Theta_2$$

of compatible implectic operators in (6.2) has a kernel which is N-dimensional ($N = 3$ in our case), and where the flow, leaving Θ_1 as well as Θ_2 invariant, has the form

$$u_t = K(u) \ . \tag{6.5}$$

We consider a basis of solutions $s_1, .., s_N$ for

$$\Theta s = (\Theta_1 - \lambda\Theta_2)s = 0 \tag{6.6}$$

and extend these over the time in the following way:

$$(s_0)_t = -K'(u)^T[s_0] \tag{6.7}$$

$$..$$

$$(s_n)_t = -K'(u)^T[s_n] + s_{n-1} \tag{6.8}$$

where $K'(u)^T$ is the transposed of the variational derivative of $K(u)$. Equation (6.7) is just the equation for the time evolution of invariant covector fields. We claim that all the s_n solve (6.6), however, now *for arbitrary* t. To see this, observe that Θ is, as a two-times contravariant tensor, invariant with respect to the flow (6.5), i.e.

$$L_K\Theta = 0$$

where the Lie-derivative of a two-times contravariant tensor with respect to the vector field G is given by:

$$L_G\Theta := \Theta'[G] - G'\Theta - \Theta G'^T$$

and where the Lie derivative of a covector field s is defined by

$$L_K s := s'[K] + K'^T[s] \ .$$

Here we just repeated the formulas we found in subsection 2.1 in coordinate representation. From this follows that the expression

$$\Theta s_0 \ ,$$

as a vector field, is invariant with respect to K:

$$\begin{aligned} L_K(\Theta s_0(t)) &= \Theta L_K(s_0(t)) \\ &= \Theta(s_0(t)_t + K'^T[s_0(t)]) \\ &= 0 \end{aligned} \qquad (6.9)$$

So invariance implies, since (6.9) is zero at one time, that $\Theta s_0(t)$ must be zero at all time. Now take the function $s_1(t)$. Then, by virtue of (6.8) and by the result just proved, we obtain

$$L_K(\Theta s_1(t)) = \Theta s_0(t) = 0 \ . \qquad (6.10)$$

Since $\Theta s_1(t)$ is zero at one time it must be zero at all times. This process can be continued to obtain the claim.

Now we define

$$\tau := s_N \ .$$

Since τ depends on u and t we express it in terms of these

$$\tau(t) = R(u,t) \ .$$

Then the t-derivatives of τ can be expressed in terms of the partial t-derivative of R and in terms of the field u (and its spatial derivatives of course). Furthermore, we express u in terms of τ

$$u = H(\tau, t)$$

and insert that in the expression obtained for the t-derivative of τ. Let us denote by R_1 and R_2 the partial t and u-derivatives of R then we obtain

$$\tau_t = R_1(H(\tau,t),t) + R_2(H(\tau,t),t)K(H(\tau,t)) \ . \qquad (6.11)$$

And since the whole procedure does not depend on a translation of the origin of t this equation does not depend explicitly on t. Thus we have now defined a flow in the field variable τ which, by (6.7) to (6.8), has the property that it acts in the kernel of Θ and has vanishing N-th derivative.
Resuming this: In the kernel of Θ, for arbitrary u, there is a nonlinear flow which, if starting with the right element, recursively generates the elements of this kernel. Furthermore, this flow is *nilpotent* because it becomes identically zero after N steps (this is how nilpotency is defined, see [16], [17]). The $(N-1)$-th derivative $s = \tau_{t\$(N-1)}$ of that flow is the gradient of the action variable we started with, and the $(N-2)$-th derivative $S = \tau_{t\$(N-2)}$, written as a covector field, fulfills

$$L_K S = s$$

hence must be the gradient of the corresponding angle variable.

So, we know that there is a nilpotent flow acting in the kernel of Θ which allows to compute a direct transformation from action to angle variables, provided these are given in diagonal form with respect to the recursion operator. This nilpotent flow we call the *action-angle equation.* For most equations one easily observes that the action-angle equation remains invariant under the transformation from τ to its $(N-1)$-th derivative

$$\tau \to s := \tau_{y\$(N-1)} \ . \tag{6.12}$$

Hence, we have completely justified our method. The only problem which remains, is to find the action-angle equation for our particular case.

It looks as if the computation of these nilpotent flows, which at the same time do factorize Θ (see [17] for that), might be rather involved. But this is far from being true. The construction of the necessary flow, indeed, can be carried out by a number of various methods, among others by the so called *squared eigenfunction relation.*

As already mentioned, for the KdV a simple analysis [14] yields that the desired nilpotent flow is

$$\tau_y = \tau D^{-1}\frac{1}{\tau} \tag{6.13}$$

or

$$\tau_{yx}\tau - \tau_y\tau_x = \tau \ . \tag{6.14}$$

For this equation the invariance (6.12) is checked directly by a simple computation. Since we already have established many links between the Korteweg de Vries equation and the Camassa-Holm equation we take this known result as the starting point for deriving the action-angle-equation for the new Camassa-Holm equation.

For this we consider the eigenfunction $\omega(x)$ of (5.4). We proceed along the following modified *variation of constants* method: We put $\tau = \tilde{\omega}^2$, then (6.14) goes over into

$$\tilde{\omega}_{xy}\tilde{\omega} - \tilde{\omega}_x\tilde{\omega}_y = 1 \tag{6.15}$$

Take a solution $\tilde{\omega}(x,y)$ of (6.15) with $\tilde{\omega}(x,y=0) = \omega(x)$ and define

$$\frac{\tilde{v}(x,y) - \tilde{v}_{xx}(x,y)}{2\lambda} + \frac{1}{4} = \frac{\tilde{\omega}_{xx}(x,y)}{\tilde{\omega}(x,y)} \ .$$

Then obviously

$$\tilde{v}(x,0) = v(x) \ .$$

Now, using (6.15) and some computation we find

$$\frac{\partial}{\partial y}\left(\frac{\tilde{v}(x,y) - \tilde{v}_{xx}(x,y)}{2\lambda} - \frac{1}{4}\right) = 0 \ .$$

Hence, the y-flow defined by (6.15) leaves (5.4) invariant. Now, setting

$$\hat{\sigma}(x,y) = (I - D^2)\tilde{\omega}(x,y)^2$$

we see that (6.14) carries over to (6.3). Integration of (6.3) leads to

$$(\hat{\sigma} - \hat{\sigma}_{xx})_y = (\hat{\sigma} - \hat{\sigma}_{xx})D^{-1}\frac{1}{(\hat{\sigma} - \hat{\sigma}_{xx})} \tag{6.16}$$

and this flow leaves invariant the kernel of the operator appearing in (6.2). So, for suitable starting values we find that $\hat{\sigma}_{yy}(x, y = 0)$ is the gradient of an action variable and $\hat{\sigma}_y(x, y = 0)$ must be the gradient of the corresponding angle variable. Finally, observing the invariance

$$\hat{\sigma} \to \sigma := \hat{\sigma}_{yy}$$

of equation (6.2) we find for

$$\sigma(x, y) := \hat{\sigma}_{yy}(x, y)$$

that now, as claimed, $\sigma(x, y = 0)$ is the gradient of an action variable and $\sigma_y(x, y = 0)$ is the gradient of the corresponding angle variable (when checking this, one has to use the nilpotency of (6.16)). This completely justifies the proposed method for finding the angle-variables.

7 The hodograph link

For most integrable equations one can easily construct the hierarchy by use of the master symmetries. These master symmetries are also strongly connected to the action-angle representation of multisolitons [13], so they are of interest on their own. In case of the Camassa-Holm equation, finding the master symmetries poses a considerable problem since the usual guesswork fails. Therefore more systematic methods must be applied to find these quantities. In order to do that, we first observe that there is a link between the Camassa-Holm equation and the KdV, at least on the formal Lie algebra level. To be precise the Camassa-Holm equation can be, by a formal hodograph link (see remark 4 below), related to

$$\left(\frac{q_{tzz}}{q_z}\right)_z + 2q_t + \left(\frac{q_t}{q}\right)_z = 0 \tag{7.1}$$

which can be considered as the first negative member of the KdV hierarchy (see (7.22)).

By a substitution of the form

$$D_x \to \rho(u) D_z \tag{7.2}$$

we can intertwine the Schrödinger operator in the following way

$$\left(D_x^2 - \frac{u}{2\lambda} - \frac{1}{4}\right) B \to A \left(D_z^2 + q(u) + \frac{k^2}{2\lambda}\right) \tag{7.3}$$

with $k =$constant, $q(u)$ to be determined and $u = v - v_{xx}$.

Comparing first orders in D_z we obtain

$$2\rho^2 B_z + \rho_z \rho B = 0\ . \tag{7.4}$$

Hence

$$B = \frac{K_1}{\sqrt{\rho}} \tag{7.5}$$

where K_1 is a constant, which, without loss of generality, is set to 1. From second order in D_z we obtain

$$A = \rho^2 B \tag{7.6}$$

and comparison of the λ-terms yields

$$u = \frac{k^2 A}{B} = k^2 \rho^2 \ . \tag{7.7}$$

Finally, the comparison of the remaining terms leads to

$$\rho^2 B_{zz} + \rho\rho_z B_z - \frac{1}{4} B = Aq = \rho^2 Bq \tag{7.8}$$

or

$$q = \frac{B_{zz}}{B} + \frac{\rho_z}{\rho}\frac{B_z}{B} - \frac{1}{4\rho^2} \ . \tag{7.9}$$

Now, using (7.4) and (7.5), i.e.

$$B = u^{-\frac{1}{4}} k^{\frac{1}{2}}$$

we can rewrite that as

$$q = \frac{k^2}{u}\left(\frac{(u^{-1/4})_{xx}}{(u^{-1/4})} - \frac{1}{4}\right) \ . \tag{7.10}$$

Observing that (7.2) is equivalent to

$$z = D_x^{-1}\rho = \frac{1}{k} D_x^{-1}\sqrt{u} \tag{7.11}$$

we see that the hodograph transformation given by (7.11) and

$$F(u) := q = \frac{k^2}{u}\left(\frac{(u^{-1/4})_{xx}}{(u^{-1/4})} - \frac{1}{4}\right) \tag{7.12}$$

transforms the modified Schrödinger eigenvector problem (5.4) into the usual one (this actually is a special case of the transformation give in [22, p. 270]).

Remark: 4 *Obviously, the hodograph transformation, as a transformation between solutions of equations, only makes sense in case that*

$$\int_{-\infty}^{+\infty} u dx \neq 0 \ .$$

However, in the following we consider it as a transformation from one abstract integro-differential algebra to another one. From this viewpoint we may consider, in any case, the hodograph transformation as a valid transformation between abstract manifolds, no matter what their analytical interpretation may be in terms of genuine solutions of differential equations. Hence we can transfer all Lie-algebra properties by it, even when a transfer of solutions is not allowed. And, in the following, these Lie algebra properties are the only subjects of our considerations.

As a consequence the isospectral flows of (5.4) are transformed into the isospectral flows for the usual Schrödinger problem and these are given by the Korteweg de Vries equation hierarchy. So, equation (2.10) (related to the Camassa-Holm equation by the Bäcklund transformation (2.11)) is by a hodograph transform carried over to a superposition of flows from the KdV-hierarchy.

Such a hodograph transform can be considered as a local manifold transform. Thus vector fields τ from the (u,x)-manifold are carried over to vector fields T on the (q,z)-manifold by (see [6]):

$$T = \left(F' - q_z D_x^{-1}\rho'\right)[\tau] =: \Pi\tau \tag{7.13}$$

where the prime denotes variational derivative with respect to u. One should observe that, because of the D_x^{-1} this transformation is only unique modulo q_z. Writing now this crucial transformation operator Π in (q,z)-coordinates we obtain

$$\Pi = -\frac{1}{4}\Phi(q)u^{-1} \tag{7.14}$$

where $\Phi(q)$ is the ordinary KdV-recursion operator (only on the (q,z)-manifold instead)

$$\Phi(\rho) = D_z^2 + 2D_z\rho D_z^{-1} + 2\rho\ . \tag{7.15}$$

Observing that

$$(D_x u D_x^{-1} + u)^{-1} = \frac{1}{2}D_x u^{-1/2} D_x^{-1} u^{-1/2} \tag{7.16}$$

we obtain Π represented by the variables on the (u,x)-manifold as

$$\Pi = \frac{k^2}{2u}(I - D_x^2)(D_x u D_x^{-1} + u)^{-1} = -\frac{k^2}{2u}\Psi^{-1}(u) \tag{7.17}$$

where Ψ is the operator from (2.9). Therefore, the transformation from the tangent bundle of the (u,x)-manifold to that of the (ρ,z)-manifold carries the recursion operator of the factorized KdV over into

$$\Psi \to \Pi\Psi(u)\Pi^{-1} = 2k^2\Phi^{-1}(q), \tag{7.18}$$

the inverse of the recursion operator of the Korteweg de Vries equation on the (q,z)-manifold. Under this transformation the image of equation (2.10) (which is a Bäcklund transformation away from the Camassa-Holm equation) is simply obtained by transforming the corresponding vector field. By this

$$u_t = \Psi(u)u_x \tag{7.19}$$

is carried over

$$q_t = \Pi\Psi(u)u_x\ . \tag{7.20}$$

Using (7.17) we see that this yields

$$q_t = -k^2\frac{u_x}{2u} \tag{7.21}$$

and because of $\Pi u_x = 0$ we find

$$\Phi(q)u_x u^{-1} = 0$$

which is

$$\Phi(q)q_t = 0 \ , \tag{7.22}$$

the first member of the negative Korteweg de Vries equation, its explicit form is (7.1). One should observe that, since T on the left side of (7.13) is only unique up to multiples of q_z, also

$$\Phi(q)q_t = f(t)\Phi(q)q_z \tag{7.23}$$

can be considered as the hodograph transform of (2.10). This reflects that we only deal with a formal Lie algebra homomorphism (isomorphism up to the generator of translation invariance) and not with a Bäcklund transformation between solutions. Again, care is advised when transforming solutions from (7.23) to those of (2.10), because for reason of nonunicity this transfer needs a subtle analysis of the special solutions involved.

So at the moment, for genuine solutions, these transformations are of a formal nature only. Still, they constitute an excellent tool in order to find other relevant invariant quantities for the original system, either for the Camassa-Holm equation or for the factorized KdV. How genuine solutions relate to this transformation still has to be investigated.

8 Master symmetries

By use of the hodograph link (considered as a transformation between integro-differential algebras) we are now able to construct the master symmetries.

Recall that

$$M_0 = \frac{1}{2}zq_z + q \tag{8.1}$$

generates a master symmetry for the Korteweg de Vries equation

$$q_t = q_{zzz} + 6q_z q \ . \tag{8.2}$$

To be precise: we have

$$L_{M_0}\Phi(q) = \Phi(q) \tag{8.3}$$

or

$$L_{M_0}\Phi(q)^{-1} = -\Phi(q)^{-1} \tag{8.4}$$

Therefore by setting

$$K_0 := q_z \tag{8.5}$$

and

$$K_{n+1} := \Phi K_n \ \text{ and } \ M_{m+1} := \Phi M_m \text{ for all } m, n \in \mathbb{N} \, .$$

we obtain the hereditary algebra (or positive indexed part of a *Virasoro Algebra* or conformal algebra) for the Korteweg de Vries equation fulfilling the following commutation relations

$$[K_n, K_m] = 0,\ [M_n, K_m] = (m+\rho)K_{n+m},\ [M_n, M_m] = (m-n)M_{n+m},$$

where $\rho := 1/2$ is a specific constant for the system under consideration (it changes for other integrable systems). Master symmetries yield important information insofar as they directly represent the action-angle structure for multisoliton solutions [13]. The knowledge of the first nontrivial master symmetry allows an efficient computation of the whole symmetry group structure of the system. Usually, if a recursion operator is known, computation of the master symmetries is simply done by starting, as it was done above, with the generator of the scaling symmetry to produce the sequence of master symmetries (or conformal symmetries as they are sometimes called). However, for the Camassa-Holm equation this approach completely fails because that equation has no obvious scaling invariance. So we have to proceed along different avenues: in this case by use of the hodograph link.

From the transformation of Lie-derivatives under manifold transformations we know that (8.3), or (8.4), by (7.18) carries over to the (u, z)-manifold as

$$L_{\Pi^{-1}M_0}\Psi(u) = -\Psi(u)\ .$$

Hence

$$\begin{aligned} m_0 : &= \Pi^{-1}M_0 = -\frac{2}{k^2}\Psi(u)uM_0 \\ &= -\frac{\Psi(u)}{k^2}\{\sqrt{u}\left(D^{-1}(\sqrt{u})\right)F(u)_x + 2uF(u)\} \end{aligned} \tag{8.6}$$

must be the scaling symmetry of the factorized KdV recursion operator, i.e.:

$$L_{m_0}\Psi(u)^{-1} = \Psi(u)^{-1}\ . \tag{8.7}$$

In this formula $F(u)$ is the quantity defined in (7.12)

$$F(u) := q = \frac{k^2}{u}\left(\frac{(u^{-1/4})_{xx}}{(u^{-1/4})} - \frac{1}{4}\right)$$

From here now the other master symmetries

$$m_n := (\Psi(u))^n m_0$$

are obtained in the usual way, and a corresponding Virasoro algebra is constructed.

References

[1] T. B. Benjamin and J. L. Bona and J. J. Mahony: *Model Equations for long Waves in Nonlinear Dispersive Systems J. Philos. Trans. Roy. Soc. London*, A 272, p.47-78, 1972

[2] J. Bona and R. Smith: *Existence of Solutions to the Korteweg-de Vries initial value problem*, in: Nonlinear Wave Motion, AMS Lectures in Applied Math. Vol.15 (A.C. Newell ed.), p.179-180, 1974

[3] R. Camassa, D.D. Holm and M. Hyman: *A Bihamiltonian Shallow water equation with Peaked Solitons*, Preprint, Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545, 1993

[4] R. Camassa, D.D. Holm and M. Hyman: *A new Integrable Shallow water equation*, Adv. Appl. Mech., 31, p.1-3, 1994

[5] R. Camassa and D.D. Holm: *An integrable shallow water equation with peaked solitons*, Phys. Rev. Letters, 71, p.1661, 1993

[6] S. Carillo and B. Fuchssteiner: *The abundant symmetry structure of hierarchies of nonlinear equations obtained by reciprocal links*, J. Math. Phys., 30, p.1606-1613, 1989

[7] B. Fuchssteiner: *Application of Hereditary Symmetries to Nonlinear Evolution equations*, Nonlinear Analysis TMA, 3, p.849-862, 1979

[8] A. S. Fokas and B. Fuchssteiner: *Bäcklund Transformations for Hereditary symmetries*, Nonlinear Analysis TMA, 5, p.423-432, 1981

[9] B. Fuchssteiner and A. S. Fokas: *Symplectic Structures, Their, Bäcklund Transformations and Hereditary Symmetries*, Physica, 4 D, p.47-66, 1981

[10] B. Fuchssteiner: *The Lie algebra structure of Degenerate Hamiltonian and Bi-Hamiltonian systems*, Progr. Theor. Phys., 68, p.1082-1104, 1982

[11] B. Fuchssteiner: *Mastersymmetries, Higher-order Time-dependent symmetries and conserved Densities of Nonlinear Evolution Equations*, Progr. Theor.Phys., 70, p.1508-1522, 1983

[12] B. Fuchssteiner: *Distribution Algebras and Elementary shock wave analysis*, in: Advances in Computer Methods for partial differential equations; IMACS 1984; eds. R. Vichnevetsky and R.S. Stepleman , 76, p.469-475, 1984

[13] B. Fuchssteiner and G. Oevel: *Geometry and action-angle variables of multisoliton systems*, Reviews in Mathematical Physics, 1, p.415-479, 1990

[14] B. Fuchssteiner and S. Carillo: *The Action-Angle transformation for Soliton Equations*, Physica, 166 A, p.651-675, 1990

[15] B. Fuchssteiner: *Hamiltonian structure and Integrability*, in: Nonlinear Systems in the Applied Sciences, Mathematics in Science and Engineering Vol. 185 Academic Press, C. Rogers and W. F. Ames eds., p.211-256, 1991

[16] B. Fuchssteiner and M. Lo Schiavo: *Nonlinear PDE's and Recursive Flows: Theory*, Appl. Math. Letters, 6, p.97-100, 1993

[17] B. Fuchssteiner and M. Lo Schiavo: *Nonlinear PDE's and Recursive Flows: Applications*, Appl. Math. Letters, 6, p.101-104, 1993

[18] I.M.Gelfand and I.Y. Dorfman: Funktsional'nyi Analiz i Ego Prilozheniya, *Hamiltonian Operators and Algebraic Structures related to them*, 13, p.13-30, 1974

[19] I.M.Gelfand and I.Y. Dorfman: Funktsional'nyi Analiz i Ego Prilozheniya, *The Schouten bracket and Hamiltonian operators*, 14, p.71-74, 1980

[20] I.M.Gelfand and I.Y. Dorfman: Funktsional'nyi Analiz i Ego Prilozheniya, *Hamiltonian Operators and Infinite-Dimensional Lie-Algebras*, 15, p.23-40, 1981

[21] A.E. Green and P.M. Naghdi: *A derivation of equations for wave propagation in water of variable depth*, J. Fluid Mech., 78, p.237-246, 1976

[22] E.L. Ince: *Ordinary differential equations*, Dover Publications, reprint, New York, 1956

[23] P. J. Olver and P. Rosenau: to appear: Phys. Rev. E, *Tri-Hamiltonian Soliton-Compacton Duality*, 1995