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A 3×3 matrix spectral problem for AKNS hierarchy and its binary Nonlinearization

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Abstract

A three-by-three matrix spectral problem for AKNS soliton hierarchy is introduced and the corresponding Bargmann symmetry constraint involving in Lax pairs and adjoint Lax pairs is discussed. An explicit new Poisson algebra is proposed and thus the liouville integrability is established for the nonlinearized spatial system and a hierarchy of nonlinearized temporal systems under the control of the nonlinearized spatial system. The obtained nonlinearized Lax systems, in which the nonlinearized spatial system is intimately related to stationary AKNS flows, lead to a sort of new involutive solutions to each AKNS soliton equation. Therefore the binary nonlinearization theory is successfully extended to a case of three-by-three matrix spectral problem for AKNS hierarchy.

Key words: Symmetry constraint, Binary nonlinearization, Involutive solution, AKNS hierarchy

1 Introduction

Symmetry constraints have aroused an increasing interest in recent few years due to the important roles they play in soliton theory. Such a kind of very successful symmetry constraint method is the nonlinearization technique for Lax pairs of soliton hierarchies, including mono-nonlinearization [1] [2] and further binary nonlinearization [3] [4].

In general, one considers the complicated nonlinear problems to be solved in such a way as breaking nonlinear problems into several linear or smaller ones and then solving these resulting problems. It is following this idea that one has introduced the method of Lax pair to study nonlinear soliton equations. The Lax pairs are always linear with respect to their eigenfunctions. Nevertheless, the nonlinearization technique puts this original object, the Lax pair, into a nonlinear and more complicated object, the nonlinearized Lax system. It doesn't seem to be reasonable enough, but in fact, it provides an effective way, different from the usual one, to solve soliton equations. The main reason why the nonlinearization technique takes effect is a kind of specific symmetry constraints expressed through the variational derivative of the spectral parameter with respect to the potential. Indeed, much of the excitement in the study of nonlinearization comes from that kind of symmetries related to the integrals of motion: the isospectral parameters, of soliton equations.

A similar symmetry constraint procedure for bi-Hamiltonian soliton hierarchies is presented by Antonowicz and Wojciechowski et al [5] [6] [7] and bi-Hamiltonian structures for the resulting classical integrable systems can also be worked out through a Miura map [6] [8]. A connection between these systems and stationary flows [9] is also given by Tondo [10] for the case of KdV hierarchy. Because stationary flows may be interpreted as finite dimensional Hamiltonian systems [9] based upon the so-called Jacobi-Ostrogradsky coordinates [11], a natural generalization of nonlinearization technique to higher order symmetry constraints is made by Zeng [12] [13] for the KdV and Kaup-Newell hierarchies etc. There have also been some algebraic geometric tricks, proposed by Flaschka et al [14] [15] [16], to deal with similar nonlinearized Lax pairs called Neumann systems.

The study of the nonlinearization theory leads to a large class of interesting finite dimensional Liouville integrable Hamiltonian systems which are connected with soliton hierarchies (for example, see [2] [17]). However in the literature, most results are presented for the cases of 2×2 matrix spectral problems. The present paper is devoted to the symmetry constraints in binary nonlinearization for a case of 3×3 matrix spectral problems. We successfully propose a 3×3 matrix spectral problem for AKNS soliton hierarchy, motivated by a representation of 3×3 matrices for the Lie algebra sl(2). Then in Section 3, we consider the Bargmann symmetry constraint for the proposed new Lax pairs and adjoint Lax pairs of AKNS soliton hierarchy and show the Liouville integrability of the resulting nonlinearized spatial system. In Section 4, we analyze the nonlinearized Lax systems, especially the nonlinearized temporal systems, and establish a sort of involutive solutions to AKNS soliton equations. Finally in Section 5, some remarks are given.

2 New Lax pairs for AKNS equations

We introduce a three-by-three matrix spectral problem

$$\phi_x = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_x = U(u,\lambda) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} -2\lambda & \sqrt{2}q & 0 \\ \sqrt{2}r & 0 & \sqrt{2}q \\ 0 & \sqrt{2}r & 2\lambda \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \qquad (2.1)$$

where the potential $u = (q, r)^T$. Its adjoint spectral problem reads as

$$\psi_x = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}_x = -U^T(u,\lambda) \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 2\lambda & -\sqrt{2}r & 0 \\ -\sqrt{2}q & 0 & -\sqrt{2}r \\ 0 & -\sqrt{2}q & -2\lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \qquad (2.2)$$

Here T means the transposition of the matrix. Our purpose is to generate AKNS hierarchy of soliton equations from the above specific spectral problem (2.1). To this end, we first solve the adjoint representation equation $V_x = [U, V]$. Take

$$V = \begin{pmatrix} 2a & \sqrt{2}b & 0 \\ \sqrt{2}c & 0 & \sqrt{2}b \\ 0 & \sqrt{2}c & -2a \end{pmatrix} = \sum_{i=0}^{\infty} \begin{pmatrix} 2a_i & \sqrt{2}b_i & 0 \\ \sqrt{2}c_i & 0 & \sqrt{2}b_i \\ 0 & \sqrt{2}c_i & -2a_i \end{pmatrix} \lambda^{-i} \tag{2.3}$$

and then we have

$$[U, V] = \begin{pmatrix} 2(qc - rb) & -2\sqrt{2}(\lambda b + qa) & 0 \\ 2\sqrt{2}(ra + \lambda c) & 0 & -2\sqrt{2}(\lambda b + qa) \\ 0 & 2\sqrt{2}(ra + \lambda c) & -2(qc - rb) \end{pmatrix}.$$

Therefore we easily find that the adjoint representation equation $V_x = [U, V]$ becomes

$$a_x = qc - rb,\ b_x = -2\lambda b - 2qa,\ c_x = 2\lambda c + 2ra,$$

which is equivalent to

$$a_{ix} = qc_i - rb_i,\ b_{ix} = -2b_{i+1} - 2qa_i,\ c_{ix} = 2c_{i+1} + 2ra_i,\ i \geq 0. \tag{2.4}$$

We fix the initial values

$$a_0 = -1,\ b_0 = c_0 = 0 \tag{2.5}$$

and require that

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0,\ i \geq 1, \tag{2.6}$$

which equivalently select constants of integration to be zero. On the other hand, the above equality (2.4) gives rise to the recursion relation for determining a_i, b_i, c_i:

$$\begin{cases} a_{i+1} = \frac{1}{2}\partial^{-1}(qc_{ix} + rb_{ix}), \\ b_{i+1} = -\frac{1}{2}b_{ix} - qa_i, \\ c_{i+1} = \frac{1}{2}c_{ix} - ra_i, \end{cases} \quad i \geq 0. \tag{2.7}$$

This recursion relation uniquely determines infinitely many sets of polynomials $a_i, b_i, c_i,\ i \geq 1$, in $u, u_x, \cdots$ under the requirement (2.6). The first two sets are as follows

$$a_1 = 0,\ b_1 = q,\ c_1 = r;\ a_2 = \frac{1}{2}(qr),\ b_2 = -\frac{1}{2}q_x,\ c_2 = \frac{1}{2}r_x.$$

In addition, we have

$$a^2 + bc = (\sum_{i=0}^{\infty} a_i\lambda^{-i})^2 + (\sum_{i=0}^{\infty} b_i\lambda^{-i})(\sum_{i=0}^{\infty} c_i\lambda^{-i}) = 1,$$

because $(a^2 + bc)_x = \frac{1}{8}\mathrm{tr}(V^2)_x = \frac{1}{8}\mathrm{tr}[U, V^2] = 0$ and $(a^2 + bc)|_{u=0} = 1$. It follows that $a_i, b_i, c_i,\ i \geq 1$, are local.

A direct computation may show that the compatibility conditions of the Lax pairs

$$\phi_x = U\phi,\ \phi_{t_n} = V^{(n)}\phi,\ V^{(n)} = V^{(n)}(u, \lambda) = (\lambda^n V)_+,\ n \geq 0, \tag{2.8}$$

or the adjoint Lax pairs

$$\psi_x = -U^T\phi,\ \psi_{t_n} = -(V^{(n)})^T\psi,\ n \geq 0, \tag{2.9}$$

where the symbol + denotes the choice of non-negative power of λ, engenders a hierarchy of AKNS soliton equations

$$u_{t_n} = \left(\begin{array}{c} q \\ r \end{array}\right)_{t_n} = K_n = \left(\begin{array}{c} -2b_{n+1} \\ 2c_{n+1} \end{array}\right) = J\Psi^n \left(\begin{array}{c} r \\ q \end{array}\right),\ n \geq 0, \tag{2.10}$$

where the Hamiltonian operator J and the recursion operator Ψ read as

$$J = \left(\begin{array}{cc} 0 & -2 \\ 2 & 0 \end{array}\right),\ \Psi = \left(\begin{array}{cc} \frac{1}{2}\partial - r\partial^{-1}q & r\partial^{-1}r \\ -q\partial^{-1}q & -\frac{1}{2}\partial + q\partial^{-1}r \end{array}\right). \tag{2.11}$$

This AKNS hierarchy is exactly the same as one in Ref. [3], which also shows that the same soliton hierarchy may possess different Lax pairs, even different order spectral matrices. Here the operator $\Phi = \Psi^*$ is a hereditary operator [18], and J and $J\Psi$ constitute a Hamiltonian pair.

Finally, we would like to elucidate the other two properties on AKNS hierarchy (2.10). First by Corollary 2.1 of Ref. [3], we can obtain

$$V_{t_n} = [V^{(n)}, V],\ n \geq 0 \tag{2.12}$$

when $u_{t_n} = K_n$, i.e. $U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0,\ n \geq 0$. Second, we can get the Hamiltonian structure of AKNS hierarchy

$$u_{t_n} = K_n = J\left(\begin{array}{c} c_{n+1} \\ b_{n+1} \end{array}\right) = J\frac{\delta H_n}{\delta u},\ H_n = \frac{2}{n+1}a_{n+2},\ n \geq 0, \tag{2.13}$$

by applying the trace identity [19] [20].

We mention that the algebraic structure hidden in the back of the above 3×3 matrix spectral problem is the same as in the case of the 2×2 one. However we shall see that there are different nonlinearized Lax systems derived from them. New resulting finite dimensional integrable systems involve more dependent variables and possess richer structures than old ones [3] .

3 Binary nonlinearization related to new spectral problem

In order to impose the Bargmann symmetry constraint in binary nonlinearization, we first need to compute the variational derivative of the spectral parameter λ with respect to the potential u, which is shown in the following Lemma [21][3].

Lemma 3.1 *Let $U(u,\lambda)$ be a matrix of order s depending on u, u_x, $\cdots$ and a parameter λ. Suppose that $\phi = (\phi_1, \phi_2, \cdots, \phi_s)^T$, $\psi = (\psi_1, \psi_2, \cdots, \psi_s)^T$ satisfy the spectral problem and the adjoint spectral problem*

$$\phi_x = U(u,\lambda)\phi, \;\; \psi_x = -U^T(u,\lambda)\psi,$$

and set the matrix $\bar{V} = \phi\psi^T = (\phi_k\psi_l)_{s\times s}$, then we have the following two results:

(i) the variational derivative of the spectral parameter λ with respect to the potential u may be expressed by

$$\frac{\delta\lambda}{\delta u} = \frac{tr(\bar{V}\frac{\partial U}{\partial u})}{-\int_{-\infty}^{\infty} tr(\bar{V}\frac{\partial U}{\partial \lambda})dx}, \tag{3.1}$$

(ii) the matrix $\bar{V}$ is a solution to the adjoint representation equation $V_x = [U, V]$, i.e. $\bar{V}_x = [U, \bar{V}]$.

Following (3.1), we have the variational derivative of the spectral parameter for the spectral problem (2.1) and the adjoint spectral problem (2.2)

$$\frac{\delta\lambda}{\delta q} = \frac{\sqrt{2}}{E}(\phi_2\psi_1 + \phi_3\psi_2), \;\; \frac{\delta\lambda}{\delta r} = \frac{\sqrt{2}}{E}(\phi_1\psi_2 + \phi_2\psi_3), \tag{3.2}$$

where $E = 2\int_{-\infty}^{\infty}(\phi_1\psi_1 - \phi_3\psi_3)dx$. The above variational derivative will serve as a conserved covariant yielding a specific symmetry used in symmetry constraints.

Let us introduce N ($N \geq 1$) distinct eigenvalues λ_j, $1 \leq j \leq N$, and denote by

$$\phi^{(j)} = (\phi_{1j}, \phi_{2j}, \phi_{3j})^T, \;\; \psi^{(j)} = (\psi_{1j}, \psi_{2j}, \psi_{3j})^T, \;\; 1 \leq j \leq N,$$

the eigenfunctions of (2.8) and the adjoint eigenfunctions of (2.9), i.e.

$$\phi^{(j)}_x = U(u,\lambda_j)\phi^{(j)}, \;\; \psi^{(j)}_x = -U^T(u,\lambda_j)\psi^{(j)}, \;\; 1 \leq j \leq N, \tag{3.3}$$

$$\phi^{(j)}_{t_n} = V^{(n)}(u,\lambda_j)\phi^{(j)}, \;\; \psi^{(j)}_{t_n} = -(V^{(n)})^T(u,\lambda_j)\psi^{(j)}, \;\; 1 \leq j \leq N. \tag{3.4}$$

Now we make the Bargmann symmetry constraint

$$K_0 = J\frac{\delta H_0}{\delta u} = J\sum_{j=1}^{N}\mu_j E_j \frac{\delta\lambda_j}{\delta u}, \tag{3.5}$$

where $E_j = 2\int_{-\infty}^{\infty}(\phi_{1j}\psi_{1j} - \phi_{3j}\psi_{3j})dx$, $1 \leq j \leq N$, and μ_j, $1 \leq j \leq N$, are any nonzero constants. Because two symmetries in the Bargmann symmetry constraint are of different types, we will see that it engenders many interesting results. By (3.2), the above symmetry constraint becomes

$$K_0 = J\sum_{j=1}^{N}\mu_j \begin{pmatrix} \sqrt{2}(\phi_{2j}\psi_{1j} + \phi_{3j}\psi_{2j}) \\ \sqrt{2}(\phi_{1j}\psi_{2j} + \phi_{2j}\psi_{3j}) \end{pmatrix},$$

from which we get the following explicit expression for the potential u

$$u = f(P_1, P_2, P_3; Q_1, Q_2, Q_3) = \sqrt{2}\begin{pmatrix} <P_1, BQ_2> + <P_2, BQ_3> \\ <P_2, BQ_1> + <P_3, BQ_2> \end{pmatrix}. \tag{3.6}$$

Here and hereafter, $<\cdot,\cdot>$ denotes the standard inner product of $\mathbb{R}^N$ and

$$B = \mathrm{diag}(\mu_1, \cdots, \mu_N), \quad \begin{pmatrix} P_i \\ Q_i \end{pmatrix} = \begin{pmatrix} (\phi_{i1}, \phi_{i2}, \cdots, \phi_{iN})^T \\ (\psi_{i1}, \psi_{i2}, \cdots, \psi_{iN})^T \end{pmatrix}, \ i = 1, 2, 3. \tag{3.7}$$

The substitution of (3.6) into the spatial system (3.3) and the temporal systems (3.4) for $n \geq 0$ yields the nonlinearized spatial system:

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_x = U(f, \lambda_j)\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \ j = 1, 2, \cdots, N, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}_x = -U^T(f, \lambda_j)\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}, \ j = 1, 2, \cdots, N; \end{cases} \tag{3.8}$$

and the nonlinearized temporal systems for $n \geq 0$:

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_{t_n} = V^{(n)}(f, \lambda_j)\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \ j = 1, 2, \cdots, N, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}_{t_n} = -(V^{(n)})^T(f, \lambda_j)\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}, \ j = 1, 2, \cdots, N. \end{cases} \tag{3.9}$$

It is obvious that (3.8) is a system of ordinary differential equations and (3.9) is a hierarchy of partial differential equations.

Suppose that Z is an expression depending on u and its differentials. From now on we use $\widetilde{Z}$ to denote the expression of Z depending on $P_i, Q_i, \ 1 \leq i \leq 3$, and their differentials after substituting (3.6) into Z, and use $\mathrm{Or}(\widetilde{Z})$ to denote the expression of $\widetilde{Z}$ only depending on $P_i, Q_i, \ 1 \leq i \leq 3$, themselves after substituting (3.8) into $\widetilde{Z}$ sufficiently many times. Therefore (3.9) may be transformed into the following systems for $n \geq 0$:

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_{t_n} = \mathrm{Or}(V^{(n)}(f, \lambda_j))\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \ j = 1, 2, \cdots, N, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}_{t_n} = -(\mathrm{Or}(V^{(n)}(f, \lambda_j))^T\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}, \ j = 1, 2, \cdots, N, \end{cases} \tag{3.10}$$

which are all ordinary differential equations with an independent variable t_n because the matrices $\mathrm{Or}(V^{(n)}(f,\lambda_j)),\ n\geq 0,\ 1\leq j\leq N$, only depend on $P_i, Q_i,\ 1\leq i\leq 3$.

We would like to discuss the integrability on the nonlinearized spatial system (3.8) and the nonlinearized temporal systems (3.10) for $n\geq 0$ in the Liouville sense [22]. We shall utilize the symplectic structure ω^2 on $\mathbb{R}^{6N}$

$$\omega^2=\sum_{i=0}^{3}\sum_{j=0}^{N}\mu_j d\phi_{ij}\wedge d\psi_{ij}=\sum_{i=0}^{3}(BdP_i)\wedge dQ_i, \tag{3.11}$$

by which one can define the corresponding Poisson bracket for two functions F, G defined over the phase space $\mathbb{R}^{6N}$

$$\begin{aligned}\{F,G\} &= \omega^2(IdG,IdF)=\omega^2(X_G,X_F)\\ &= \sum_{i=1}^{3}\sum_{j=1}^{N}\mu_j^{-1}(\frac{\partial F}{\partial\psi_{ij}}\frac{\partial G}{\partial\phi_{ij}}-\frac{\partial F}{\partial\phi_{ij}}\frac{\partial G}{\partial\psi_{ij}})\\ &= \sum_{i=1}^{3}(<\frac{\partial F}{\partial Q_i},B^{-1}\frac{\partial G}{\partial P_i}>-<\frac{\partial F}{\partial P_i},B^{-1}\frac{\partial G}{\partial Q_i}>),\end{aligned} \tag{3.12}$$

where $IdH=X_H$ denotes the Hamiltonian vector field with energy H determined by

$$\omega^2(X,IdH)=\omega^2(X,X_H)=dH(X),\ X\in T(\mathbb{R}^{6N}),$$

and the corresponding Hamiltonian system with the Hamiltonian function H

$$\dot{x}=IdH(x)=dx(IdH)=\omega^2\{IdH,Idx\}=\{x,H\},\ x\in\mathbb{R}^{6N}, \tag{3.13}$$

which possesses an explicit formulation

$$\dot{P}_i=-B^{-1}\frac{\partial H}{\partial Q_i},\ \dot{Q}_i=B^{-1}\frac{\partial H}{\partial P_i},\ i=1,2,3. \tag{3.14}$$

Note that there are some authors who use the other Poisson bracket $\{F,G\} = \omega^2(X_F,X_G)$. As remarked by Carroll [23], it doesn't matter of course but each type has many proponents and hence one must be careful of minus signs in reading various sources. The notation we accept here is the Arnold's one [24].

Theorem 3.1 *The following functions*

$$\bar{F}_j=\sum_{i=1}^{3}\phi_{ij}\psi_{ij},\ 1\leq j\leq N, \tag{3.15}$$

are all integrals of motion for the nonlinearized spatial system (3.8). Moreover they are in involution under the Poisson bracket (3.12) and independent over the region

$$\Omega=\{\mathbb{R}^{6N}|\,\phi_{ij},\psi_{ij}\in\mathbb{R},\ \sum_{i=1}^{3}(\phi_{ij}^2+\psi_{ij}^2)\neq 0,\ 1\leq j\leq N\}.$$

Proof: Let

$$\bar{V}(\lambda_j) = (\phi_{kj}\psi_{lj})_{k,l=1,2,3}\,,\ 1 \le j \le N.$$

We can first find that

$$\bar{F}_j = \mathrm{tr}(\bar{V}(\lambda_j)).$$

On the other hand, by Lemma 3.1 we know that $\bar{V}(\lambda_j)$ satisfies

$$\bar{V}(\lambda_j)_x = [U(\tilde{u}, \lambda_j), \bar{V}(\lambda_j)]$$

when (3.8) holds, and thus

$$\begin{array}{l} \bar{F}_{jx} = (\mathrm{tr}(\bar{V}(\lambda_j)))_x = \mathrm{tr}(\bar{V}(\lambda_j))_x \\ = \mathrm{tr}[U(\tilde{u}, \lambda_j), \bar{V}(\lambda_j)] = 0, \end{array}$$

which shows that $\bar{F}_j$, $1 \le j \le N$, are all integrals of motion for the nonlinearized spatial system (3.8). In addition, it is very easy to prove that

$$\{\bar{F}_k, \bar{F}_l\} = 0,\ 1 \le k, l \le N,$$

which means $\bar{F}_j,\ 1 \le j \le N$, are in involution. It is also obvious that grad$\bar{F}_j$, $1 \le j \le N$, are everywhere functionally independent over Ω by observing that

$$\begin{array}{lcl} \left(\frac{\partial \bar{F}_k}{\partial \phi_{il}} \right)_{k,l=1,\cdots,N} & = & \begin{pmatrix} \psi_{i1} & & & 0 \\ & \psi_{i2} & & \\ & & \ddots & \\ 0 & & & \psi_{iN} \end{pmatrix},\ i = 1, 2, 3, \\ \left(\frac{\partial \bar{F}_k}{\partial \psi_{il}} \right)_{k,l=1,\cdots,N} & = & \begin{pmatrix} \phi_{i1} & & & 0 \\ & \phi_{i2} & & \\ & & \ddots & \\ 0 & & & \phi_{iN} \end{pmatrix},\ i = 1, 2, 3. \end{array}$$

The proof is completed. ▮

Throughout our paper, we assume that

$$A = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N). \qquad (3.16)$$

If all elements $Z_{ij},\ 1 \le i, j \le s$ of a given matrix $Z = (Z_{ij})_{s\times s}$ are polynomials in λ, i. e. $Z_{ij} = \sum_{k=0}^{m} Z_{ijk}\lambda^k$, for convenience of presentation, we define a new matrix called $M_A(Z)$ as follows

$$M_A(Z) = (\sum_{k=0}^{m} Z_{ijk}A^k)_{sN\times sN}. \qquad (3.17)$$

Moreover we often adopt compact forms, for example,

$$\frac{\partial}{\partial P_i} = (\frac{\partial}{\partial \phi_{i1}}, \cdots, \frac{\partial}{\partial \phi_{iN}})^T,\ \{P_i, H\} = (\{\phi_{i1}, H\}, \cdots, \{\phi_{iN}, H\})^T,\ i = 1, 2, 3.$$

Theorem 3.2 *We have the explicit integrals of motion for the nonlinearized spatial system (3.8):*

$$\begin{cases} F_1 = -8(< P_1, BQ_1 > - < P_3, BQ_3 >), \\ F_m = 4\sum_{i=1}^{m-1}[(< A^{i-1}P_1, BQ_1 > - < A^{i-1}P_3, BQ_3 >)\times \\ \qquad (< A^{m-i-1}P_1, BQ_1 > - < A^{m-i-1}P_3, BQ_3 >) \\ \qquad +2(< A^{i-1}P_1, BQ_2 > + < A^{i-1}P_2, BQ_3 >)\times \\ \qquad (< A^{m-i-1}P_2, BQ_1 > + < A^{m-i-1}P_3, BQ_2 >)] \\ \qquad -8(< A^{m-1}P_1, BQ_1 > - < A^{m-1}P_3, BQ_3 >), \ m \geq 2, \end{cases} \tag{3.18}$$

where P_i, Q_i, B *are defined by (3.7). Moreover they constitute an involutive system together with* $\bar{F}_j$, $1 \leq j \leq N$, *under the Poisson bracket (3.12), i.e.*

$$\{F_k, F_l\} = \{F_m, \bar{F}_j\} = 0, \ m, k, l \geq 1, \ 1 \leq j \leq N.$$

Proof: We assume that

$$\hat{q} = \sqrt{2}(< P_1, BQ_2 > + < P_2, BQ_3 >), \ \hat{r} = \sqrt{2}(< P_2, BQ_1 > + < P_3, BQ_2 >);$$
$$\hat{a}_0 = -1, \ \hat{b}_0 = \hat{c}_0 = 0;$$
$$\begin{cases} \hat{a}_{i+1} =< A^iP_1, BQ_1 > - < A^iP_3, BQ_3 >, \ i \geq 0, \\ \hat{b}_{i+1} = \sqrt{2}(< A^iP_1, BQ_2 > + < A^iP_2, BQ_3 >), \ i \geq 0, \\ \hat{c}_{i+1} = \sqrt{2}(< A^iP_2, BQ_1 > + < A^iP_3, BQ_2 >), \ i \geq 0. \end{cases} \tag{3.19}$$

Further we choose that

$$\hat{U} = \begin{pmatrix} -2\lambda & \sqrt{2}\hat{q} & 0 \\ \sqrt{2}\hat{r} & 0 & \sqrt{2}\hat{q} \\ 0 & \sqrt{2}\hat{r} & 2\lambda \end{pmatrix}, \ \hat{V} = \begin{pmatrix} 2\hat{a} & \sqrt{2}\hat{b} & 0 \\ \sqrt{2}\hat{c} & 0 & \sqrt{2}\hat{b} \\ 0 & \sqrt{2}\hat{c} & -2\hat{a} \end{pmatrix}, \tag{3.20}$$

where $\hat{a}$, $\hat{b}$ and $\hat{c}$ are defined by

$$\hat{a} = \sum_{i=0}^{\infty} \hat{a}_i\lambda^{-i}, \ \hat{b} = \sum_{i=0}^{\infty} \hat{b}_i\lambda^{-i}, \ \hat{c} = \sum_{i=0}^{\infty} \hat{c}_i\lambda^{-i}.$$

It may be shown that when the nonlinearized spatial system (3.8) holds, we have

$$\hat{V}_x = [\hat{U}, \hat{V}], \ \text{i.e.} \ \hat{a}_x = \hat{q}\hat{c} - \hat{r}\hat{b}, \ \hat{b}_x = -2\lambda\hat{b} - 2\hat{q}\hat{a}, \ \hat{c}_x = 2\lambda\hat{c} + 2\hat{r}\hat{a}.$$

Therefore we can compute that

$$\hat{F}_x := (\frac{1}{2}\mathrm{tr}(\hat{V}^2))_x = \frac{1}{2}\mathrm{tr}(\hat{V}^2)_x = \frac{1}{2}\mathrm{tr}[\hat{U}, \hat{V}^2] = 0.$$

On the other hand, we have

$$\hat{F} = 4(\hat{a}^2 + \hat{b}\hat{c}) = \sum_{m=0}^{\infty} F_m\lambda^{-m}, \ F_0 = 4, \ F_m = 4\sum_{i=0}^{m}(\hat{a}_i\hat{a}_{m-i} + \hat{b}_i\hat{c}_{m-i}), \ m \geq 1.$$

Hence $F_m,\ m \geq 1$, are all integrals of motion for the nonlinearized spatial system (3.8).

Now we turn to the involutivity of integrals of motion. We take

$$\hat{V}^{(n)}(\lambda) = (\lambda^n \hat{V})_+$$

and construct a temporal system for $n \geq 0$

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}_{t_n} = M_A(\hat{V}^{(n)}) \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}, \quad \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}_{t_n} = -(M_A(\hat{V}^{(n)}))^T \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}, \tag{3.21}$$

where $M_A(\hat{V}^{(n)})$, $n \geq 0$, are determined in the way as (3.17). We can first prove that when this system (3.21) holds, we have

$$(\hat{V}(\lambda))_{t_n} = [\hat{V}^{(n)}(\lambda), \hat{V}(\lambda)].$$

Therefore $F_m,\ m \geq 1$, are also integrals of motion for the system (3.21). Secondly, we can verify that

$$\begin{pmatrix} B^{-1}\frac{\partial \hat{F}}{\partial P_1} \\ B^{-1}\frac{\partial \hat{F}}{\partial P_2} \\ B^{-1}\frac{\partial \hat{F}}{\partial P_3} \end{pmatrix} = \begin{pmatrix} B^{-1}\mathrm{tr}(\hat{V}\frac{\partial}{\partial P_1}\hat{V}) \\ B^{-1}\mathrm{tr}(\hat{V}\frac{\partial}{\partial P_2}\hat{V}) \\ B^{-1}\mathrm{tr}(\hat{V}\frac{\partial}{\partial P_3}\hat{V}) \end{pmatrix} = \sum_{m=0}^{\infty} (M_A(\hat{V}^{(m)}))^T \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} \lambda^{-m-1},$$

$$\begin{pmatrix} B^{-1}\frac{\partial \hat{F}}{\partial Q_1} \\ B^{-1}\frac{\partial \hat{F}}{\partial Q_2} \\ B^{-1}\frac{\partial \hat{F}}{\partial Q_3} \end{pmatrix} = \begin{pmatrix} B^{-1}\mathrm{tr}(\hat{V}\frac{\partial}{\partial Q_1}\hat{V}) \\ B^{-1}\mathrm{tr}(\hat{V}\frac{\partial}{\partial Q_2}\hat{V}) \\ B^{-1}\mathrm{tr}(\hat{V}\frac{\partial}{\partial Q_3}\hat{V}) \end{pmatrix} = \sum_{m=0}^{\infty} (M_A(\hat{V}^{(m)})) \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} \lambda^{-m-1}.$$

These two equalities show that the system (3.21) for $n \geq 0$ are all Hamiltonian systems with Hamiltonian functions $-F_{n+1}$. Therefore

$$\{F_{m+1}, -F_{n+1}\} = \frac{d}{dt_n} F_{m+1} = 0,\ m, n \geq 0,$$

which shows the involutivity of F_m, $m \geq 1$. In addition, it is easy to get that

$$\{F_k, \bar{F}_l\} = \sum_{i=0}^{3} \mu_l^{-1} (\frac{\partial F_k}{\partial \psi_{il}} \psi_{il} - \frac{\partial F_k}{\partial \phi_{il}} \phi_{il}) = 0,\ k \geq 1,\ 1 \leq l \leq N,$$

noting the particular form of F_k, $k \geq 1$. The proof is finished. ▮

Because we have $\bar{V}^2(\lambda_j) = \mathrm{tr}(\bar{V}(\lambda_j))\bar{V}(\lambda_j)$, $1 \leq j \leq N$, $\hat{V}^3 = \frac{1}{2}\mathrm{tr}(\hat{V}^2)\hat{V}$, we cannot obtain new integrals of motion of the nonlinearized spatial system (3.8) from the trace of other power of $\bar{V}(\lambda_j)$ and $\hat{V}$. It is also interesting to observe that the determinants of the matrices $\bar{V}(\lambda_j)$, $1 \leq j \leq N$, and $\hat{V}$ are all zero.

Note that the above proof provides, in fact, us a way to construct involutive integrals of motion of the nonlinearized spatial system through the adjoint representation equation. To calculate integrals of motion, we have assumed that the potential u vanishes at infinity and thus we have a simple formula $\Psi(\delta\lambda/\delta u) = \lambda(\delta\lambda/\delta u)$ leading to explicit expressions $\hat{a}_i,\ \hat{b}_i,\ \hat{c}_i,\ i \ge 0$, of the functions $a_i,\ b_i,\ c_i,\ i \ge 0$, under the symmetry constraint (3.5) and the nonlinearized spatial system (3.8). This formula doesn't, in general, hold for the potential with non-zero boundary condition. Therefore precise expressions of various quantities need to add some extra terms. We will see this in the next section.

The proof of the above theorem also gives rise to a kind of Lax representations of the nonlinearized Lax systems. It is easy to see that $\hat{a}$, $\hat{b}$, $\hat{c}$ have the compact forms

$$\begin{aligned} \hat{a} &= \sum_{j=1}^{N} \frac{\lambda_j \mu_j}{\lambda - \lambda_j}(\phi_{1j}\psi_{1j} - \phi_{3j}\psi_{3j}) - 1, \\ \hat{b} &= \sqrt{2}\sum_{j=1}^{N} \frac{\lambda_j \mu_j}{\lambda - \lambda_j}(\phi_{1j}\psi_{2j} + \phi_{2j}\psi_{3j}), \\ \hat{c} &= \sqrt{2}\sum_{j=1}^{N} \frac{\lambda_j \mu_j}{\lambda - \lambda_j}(\phi_{2j}\psi_{1j} + \phi_{3j}\psi_{2j}), \end{aligned}$$

which lead to an expression of finite form for $\hat{V}$. Therefore we obtain a Lax representation of the nonlinearized spatial system (3.8): $\hat{V}_x = [\hat{U}, \hat{V}]$ where $\hat{U}$, $\hat{V}$ defined by (3.20), and Lax representations of the nonlinearized temporal systems (3.10): $\hat{V}_{t_n} = [\hat{V}^{(n)}, \hat{V}]$ where $\hat{V}^{(n)} = (\lambda^n \hat{V})_+$. Strictly speaking, this kind of Lax representations are only necessary but not sufficient. The reason is that we haven't an injective Gateaux derivative operator of $\hat{V}$, namely that we can't get zero vector field $K = (K_1, \cdots, K_{6N})^T = 0$ from $\hat{V}'[K] = 0$. Moreover Lax representations $\hat{V}_{t_n} = [\hat{V}^{(n)}, \hat{V}]$ of the nonlinearized temporal systems (3.10) hold only under the zero boundary condition, because the nonlinearized temporal systems (3.10) are equivalently reduced to (3.21) under the zero boundary condition. However this kind of necessary Lax representations has extensively been considered [25] [26], and used for constructing r-matrices (sometimes possibly dynamical) and canonically conjugate variables [27] [28] and for considering separation of variables [29] [28] of the resulting finite dimensional Hamiltonian systems.

The following theorem is important to show the integrability of the nonlinearized Lax systems in the Liouville sense.

Theorem 3.3 *The functions $\bar{F}_k,\ 1 \le k \le N,\ F_m,\ 1 \le m \le 2N$, are functionally independent over some region of $\mathbb{R}^{6N}$ and thus constitute an involutive system of independent functions over some region of $\mathbb{R}^{6N}$.*

Proof: Suppose that the result of the theorem is not true, that is to say, there doesn't exist any region of $\mathbb{R}^{6N}$ over which the functions $\bar{F}_k,\ 1 \le k \le N,\ F_m,\ 1 \le$

$m \leq 2N$, can be functionally independent. Therefore there exist $3N$ constants α_k, $1 \leq k \leq N,\ \beta_m,\ 1 \leq m \leq 2N$, satisfying

$$\sum_{k=1}^{N} \alpha_k^2 + \sum_{m=1}^{2N} \beta_m^2 \neq 0, \tag{3.22}$$

so that we have for all points in $\mathbb{R}^{6N}$

$$\sum_{k=1}^{N} \alpha_k \left(\left(\frac{\partial \bar{F}_k}{\partial P_1}\right)^T, \left(\frac{\partial \bar{F}_k}{\partial P_2}\right)^T, \left(\frac{\partial \bar{F}_k}{\partial P_3}\right)^T \right) + \sum_{m=1}^{2N} \beta_m \left(\left(\frac{\partial F_m}{\partial P_1}\right)^T, \left(\frac{\partial F_m}{\partial P_2}\right)^T, \left(\frac{\partial F_m}{\partial P_3}\right)^T \right) = 0. \tag{3.23}$$

In order to arise a contradiction, we will utilize the following equalities, which may directly be worked out,

$$\begin{aligned}
\frac{\partial \bar{F}_k}{\partial P_i}\Big|_{P_1=P_3=0} &= (0, \cdots, 0, \overset{k}{\psi_{ik}}, 0, \cdots, 0)^T,\ i = 1, 2, 3, \\
\frac{\partial F_m}{\partial P_1}\Big|_{P_1=P_3=0} &= 8 \textstyle\sum_{i=1}^{m-1} < A^{m-i-1}P_2, BQ_1 > A^{i-1}BQ_2 - 8A^{m-1}BQ_1, \\
\frac{\partial F_m}{\partial P_2}\Big|_{P_1=P_3=0} &= 8 \textstyle\sum_{i=1}^{m-1} (< A^{m-i-1}P_2, BQ_1 > A^{i-1}BQ_3 + < A^{i-1}P_2, BQ_3 > A^{m-i-1}BQ_1), \\
\frac{\partial F_m}{\partial P_3}\Big|_{P_1=P_3=0} &= 8 \textstyle\sum_{i=1}^{m-1} < A^{i-1}P_2, BQ_3 > A^{m-i-1}BQ_2 + 8A^{m-1}BQ_3.
\end{aligned}$$

Here $1 \leq k \leq N,\ 1 \leq m \leq 2N$ and we accept that the sum terms take zero value when $m = 1$. First of all, the equality (3.23) upon choosing $P_1 = P_3 = Q_1 = Q_3 = 0$ leads to

$$(0, \cdots, 0, \alpha_1\psi_{21}, \cdots, \alpha_N\psi_{2N}, 0, \cdots, 0)^T = 0.$$

This means $\alpha_1, \cdots, \alpha_N$ must equal to zero and thus

$$\sum_{m=1}^{2N} \beta_m \left(\left(\frac{\partial F_m}{\partial P_1}\right)^T, \left(\frac{\partial F_m}{\partial P_2}\right)^T, \left(\frac{\partial F_m}{\partial P_3}\right)^T \right)\Bigg|_{P_1=P_3=0} = 0. \tag{3.24}$$

And then we choose $P_2 = 0$. The above equality (3.24) yields

$$\sum_{m=1}^{2N} \beta_m \lambda_j^{m-1} = 0,\ 1 \leq j \leq N. \tag{3.25}$$

Now after choosing $Q_2 = 0,\ Q_1 = Q_3$, the equality (3.24) can give rise to

$$\sum_{m=2}^{2N} (m-1)\beta_m \lambda_j^{m-2} = 0,\ 1 \leq j \leq N. \tag{3.26}$$

We observe the system of algebraic equations: (3.25) and (3.26). To the end, we define a polynomial $P(\lambda)$ by

$$P(\lambda) = \sum_{m=1}^{2N} \beta_m \lambda^{m-1}.$$

The equalities (3.25) and (3.26) mean that $P(\lambda_j)=0,\ P'(\lambda_j)=0,\ 1\le j\le N$, which implies $P(\lambda)$ has $2N$ roots. But $P(\lambda)$ is only a polynomial with degree $2N-1$ and thus it must be zero. So we have $\beta_m=0,\ 1\le m\le 2N$. In fact, by the mathematical induction we can give a formula for the coefficient determinant, denoted by $C(N)$, of (3.25) and (3.26)

$$C(N)=\left|\begin{array}{ccccc} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{2N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^{2N-1} \\ 0 & 1 & 2\lambda_1 & \cdots & (2N-1)\lambda_1^{2N-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 2\lambda_N & \cdots & (2N-1)\lambda_N^{2N-2} \end{array}\right| = (-1)^{\frac{1}{2}N(N-1)}\prod_{1\le i<j\le N}(\lambda_i-\lambda_j)^4,$$

because we have the relation

$$C(N)=(-1)^{N-1}\lambda_N^{4(N-1)}C(N-1)+\text{less order terms of }\lambda_N,$$

by Laplace expansion for N-th and $2N$-th rows, and the equalities

$$\frac{d^iC(N)}{d\lambda_N^i}\,|_{\lambda_N=\lambda_j}=0,\ 1\le i\le 3,\ 1\le j\le N-1.$$

Therefore we finally obtain all zero constants: $\alpha_k=0,\ 1\le k\le N,\ \beta_m=0,\ 1\le m\le 2N$. This contradicts to the statement at the beginning of the proof. Therefore the functions $\bar F_k,\ 1\le k\le N,\ F_m,\ 1\le m\le 2N$, may be functionally independent at least on certain region of $\mathbb{R}^{6N}$. The proof is finished. ▮

The independence of integrals of motion may also be shown by computer algebra system. For example, we can calculate that

$$\begin{array}{rcl} D(1) &=& (16\phi_{11}\psi_{11}\psi_{21}^3+16\phi_{31}\psi_{21}^3\psi_{31}-32\phi_{11}\psi_{11}^2\psi_{21}\psi_{31} \\ && +32\phi_{21}\psi_{11}\psi_{21}^2\psi_{31}-32\phi_{31}\psi_{11}\psi_{21}\psi_{31}^2-64\phi_{21}\psi_{11}^2\psi_{31}^2)\mu_1^3 \end{array}$$

and that

$$D(2)=(-256\lambda_1^3+256\lambda_2^3-768\lambda_1\lambda_2^2+768\lambda_1^2\lambda_2)\mu_1^3\mu_2^4,$$

when we choose

$$\begin{array}{l} \phi_{11}=1,\ \phi_{12}=0,\ \phi_{21}=0,\ \phi_{22}=-1,\ \phi_{31}=0,\ \phi_{32}=0, \\ \psi_{11}=0,\ \psi_{12}=-1,\ \psi_{21}=1,\ \psi_{22}=0,\ \psi_{31}=0,\ \psi_{32}=-1. \end{array}$$

Here $D(N)$ is defined by

$$D(N)=\left|\begin{array}{ccc} (\mathrm{grad}_{P_1}\bar F_1)^T & (\mathrm{grad}_{P_2}\bar F_1)^T & (\mathrm{grad}_{P_3}\bar F_1)^T \\ \vdots & \vdots & \vdots \\ (\mathrm{grad}_{P_1}\bar F_N)^T & (\mathrm{grad}_{P_2}\bar F_N)^T & (\mathrm{grad}_{P_3}\bar F_N)^T \\ (\mathrm{grad}_{P_1}F_1)^T & (\mathrm{grad}_{P_2}F_1)^T & (\mathrm{grad}_{P_3}F_1)^T \\ \vdots & \vdots & \vdots \\ (\mathrm{grad}_{P_1}F_{2N})^T & (\mathrm{grad}_{P_2}F_{2N})^T & (\mathrm{grad}_{P_3}F_{2N})^T \end{array}\right|$$

with $\text{grad}_{P_i} G = \frac{\partial G}{\partial P_i}$, $i = 1, 2, 3$. These are consequences of computation by the computer algebra system MuPAD [30]. More generally, we can similarly show the involutivity of the functions $\bar{F}_k,\ 1 \le k \le N,\ F_{i_1}, \cdots, F_{i_{2N}}$, where $i_1, \cdots, i_{2N}$ are $2N$ distinct integers.

Let us now give a result about the integrability for the nonlinearized spatial system (3.8), which now needs just a direct computation.

Theorem 3.4 *The nonlinearized spatial system (3.8) may be rewritten as a Liouville integrable Hamiltonian system*

$$P_{ix} = \{P_i, H\} = -B^{-1}\frac{\partial H}{\partial Q_i},\ Q_{ix} = \{Q_i, H\} = B^{-1}\frac{\partial H}{\partial P_i},\ i = 1, 2, 3 \tag{3.27}$$

with the Hamiltonian function

$$\begin{aligned} &H = 2(< AP_1, BQ_1 > - < AP_3, BQ_3 >) \\ &-2(< P_1, BQ_2 > + < P_2, BQ_3 >)(< P_2, BQ_1 > + < P_3, BQ_2 >) = -\tfrac{1}{4}F_2 + \tfrac{1}{64}F_1^2. \end{aligned}$$

Thus it possesses a hierarchy of involutive integrals of motion $\bar{F}_j,\ 1 \le j \le N, F_m,\ m \ge 1$, *of which* $\bar{F}_k,\ 1 \le k \le N,\ F_m,\ 1 \le m \le 2N$, *are functionally independent.*

We point out that our integrable system (3.27) and involutive system (3.18) are all completely different from ones related to the original 2×2 Lax pairs [3]. It seems to us that there aren't any relation between them. The existence of various associated finite dimensional integrable systems also shows the diversity of solutions to AKNS equations. The symmetry constraints (3.5) considered here is a general linear combination of the specific symmetries $J(\delta\lambda_j/\delta u),\ 1 \le j \le N$, and thus the resulting nonlinearized Lax systems correspond to a non-standard canonical structure (3.11).

4 Involutive solutions

The aim of this section is to further discuss some properties on the nonlinearized spatial system (3.8) and the nonlinearized spatial system (3.10) and to establish a kind of involutive solutions with separated variables for AKNS soliton equations.

Lemma 4.1 *When* $\phi = (\phi_1, \phi_2, \phi_3)^T$ *and* $\psi = (\psi_1, \psi_2, \psi_3)^T$ *satisfy the spectral problem (2.1) and the adjoint spectral problem (2.2), we have*

$$\Psi \begin{pmatrix} \phi_2\psi_1 + \phi_3\psi_2 \\ \phi_1\psi_2 + \phi_2\psi_3 \end{pmatrix} = \lambda \begin{pmatrix} \phi_2\psi_1 + \phi_3\psi_2 \\ \phi_1\psi_2 + \phi_2\psi_3 \end{pmatrix} + I \begin{pmatrix} r \\ q \end{pmatrix}, \tag{4.1}$$

where Ψ *is defined by (2.11) and* I *is an integral of motion for (2.1) and (2.2).*

Proof: From the spectral problem (2.1) and the adjoint spectral problem (2.2), we can find that

$$\partial(\phi_1\psi_1-\phi_3\psi_3)=\sqrt{2}q(\phi_2\psi_1+\phi_3\psi_2)-\sqrt{2}r(\phi_1\psi_2+\phi_2\psi_3).$$

This yields

$$\partial^{-1}[-q(\phi_2\psi_1+\phi_3\psi_2)+r(\phi_1\psi_2+\phi_2\psi_3)]=-\frac{1}{\sqrt{2}}(\phi_1\psi_1-\phi_3\psi_3)+I,$$

where I is an integrals of motion for (2.1) and (2.2). The relation (4.1) follows from the above equality. ▌

We recall that $\widetilde{Z}$ denotes the expression of Z depending on $P_i, Q_i,\ 1\le i\le 3$, and their differentials after the substitution of (3.6) and into Z, and that $\mathrm{Or}(\widetilde{Z})$ denotes the expression of $\widetilde{Z}$ only depending on $P_i, Q_i,\ 1\le i\le 3$, themselves after the substitution of (3.8) into $\widetilde{Z}$ sufficiently many times. A general result on $\widetilde{a}_i, \widetilde{b}_i, \widetilde{c}_i,\ i\ge 1$, is given in the following theorem.

Theorem 4.1 *We have the explicit expressions for $\widetilde{a}_m,\ \widetilde{b}_m,\ \widetilde{c}_m, m\ge 1$:*

$$\widetilde{a}_{m+1} = \sum_{i=0}^{m} I_i(< A^{m-i}P_1, BQ_1 > - < A^{m-i}P_3, BQ_3 >) - I_{m+1},\ m\ge 0, \quad (4.2)$$

$$\widetilde{b}_{m+1} = \sqrt{2}\sum_{i=0}^{m} I_i(< A^{m-i}P_1, BQ_2 > + < A^{m-i}P_2, BQ_3 >),\ m\ge 0, \quad (4.3)$$

$$\widetilde{c}_{m+1} = \sqrt{2}\sum_{i=0}^{m} I_i(< A^{m-i}P_2, BQ_1 > + < A^{m-i}P_3, BQ_2 >),\ m\ge 0, \quad (4.4)$$

provided that the nonlinearized spatial system (3.8) is satisfied. Here $I_m,\ m\ge 0$, are defined by

$$I_0=1,\ I_m=\sum_{n=1}^{m} d_n \sum_{\substack{i_1+\cdots+i_n=m\\ i_1,\cdots,i_n\ge 1}} F_{i_1}\cdots F_{i_n},\ m\ge 1, \quad (4.5)$$

where the constants $d_n,\ n\ge 0$, are determined recursively by

$$\begin{cases} d_1=-\frac{1}{8},\ d_2=\frac{3}{128}, \\ d_n=-\frac{1}{2}\sum_{i=1}^{n-1} d_i d_{n-i}-\frac{1}{4}d_{n-1}-\frac{1}{8}\sum_{i=1}^{n-2} d_i d_{n-i-1},\ n\ge 3, \end{cases} \quad (4.6)$$

and the functions $F_m,\ m\ge 1$, are given by (3.18).

Proof: By the recursion relation (2.7) and Lemma (4.1), we can obtain that

$$\begin{pmatrix} \widetilde{c}_{m+1} \\ \widetilde{b}_{m+1} \end{pmatrix} = \widetilde{L}^m \begin{pmatrix} \widetilde{r} \\ \widetilde{q} \end{pmatrix} = \widetilde{L}^m \begin{pmatrix} \sqrt{2}(< P_2, BQ_1 > + < P_3, BQ_2 >) \\ \sqrt{2}(< P_1, BQ_2 > + < P_2, BQ_3 >) \end{pmatrix}$$

$$= \sum_{i=0}^{m} I_i \begin{pmatrix} \sqrt{2}(< A^{m-i}P_2, BQ_1 > + < A^{m-i}P_3, BQ_2 >) \\ \sqrt{2}(< A^{m-i}P_1, BQ_2 > + < A^{m-i}P_2, BQ_3 >) \end{pmatrix},\ m\ge 0,$$

where $I_0 = 1$ and $I_i,\ 1 \le i \le m$, are integrals of motion for the nonlinearized spatial system (3.8). Now we compute $\widetilde{a}_{m+1},\ m \ge 0$, by $\widetilde{b}_{mx} = -2\widetilde{b}_{m+1} - 2\widetilde{q}\,\widetilde{a}_m,\ m \ge 0$. Noting (3.8), we have for $m \ge 0$

$$\begin{aligned}
\widetilde{b}_{mx} &= \sum_{i=0}^{m-1} \sqrt{2} I_i (< A^{m-i-1}P_{1x}, BQ_2 > + < A^{m-i-1}P_1, BQ_{2x} > \\
&\qquad + < A^{m-i-1}P_{2x}, BQ_3 > + < A^{m-i-1}P_2, BQ_{3x} >) \\
&= \sum_{i=0}^{m-1} \sqrt{2} I_i (< -2A^{m-i}P_1, BQ_2 > + < A^{m-i-1}P_1, -\sqrt{2}\widetilde{q}BQ_1 > \\
&\qquad + < \sqrt{2}\widetilde{q}A^{m-i-1}P_3, BQ_3 > + < A^{m-i-1}P_2, -2ABQ_3 >) \\
&= -2\widetilde{b}_{m+1} - 2\widetilde{q}\Big(\sum_{i=0}^{m-1} I_i (< A^{m-i-1}P_1, BQ_1 > - < A^{m-i-1}P_3, BQ_3 >) - I_m\Big),
\end{aligned}$$

from which (4.2) follows.

In the following we determine the integrals of motion $I_m,\ m \ge 0$, by a relation

$$\widetilde{a}^2 + \widetilde{b}\widetilde{c} = 1,\ \widetilde{a} = \sum_{i=0}^{\infty} \widetilde{a}_i \lambda^{-i},\ \widetilde{b} = \sum_{i=0}^{\infty} \widetilde{b}_i \lambda^{-i},\ \widetilde{c} = \sum_{i=0}^{\infty} \widetilde{c}_i \lambda^{-i},$$

which gives rise to

$$2\widetilde{a}_m = \sum_{i=1}^{m-1} (\widetilde{a}_i \widetilde{a}_{m-i} + \widetilde{b}_i \widetilde{c}_{m-i}),\ m \ge 2. \tag{4.7}$$

First from $\widetilde{a}_1 = 0$ we have

$$I_1 = < P_1, BQ_1 > - < P_3, BQ_3 > = -\frac{1}{8} F_1,$$

which shows $d_1 = -\frac{1}{8}$. Now we suppose $m \ge 2$. At this moment, we have by (4.7)

$$\begin{aligned}
&2\sum_{i=0}^{m-1} I_i \hat{a}_{m-i} - 2I_m \\
&= \sum_{i=1}^{m-1} (\sum_{k=0}^{i-1} I_k \hat{a}_{i-k} - I_i)(\sum_{l=0}^{m-i-1} I_l \hat{a}_{m-i-l} - I_{m-i}) \\
&\quad + \sum_{i=1}^{m-1} \sum_{k=0}^{i-1} I_k \hat{b}_{i-k} \sum_{l=0}^{m-i-1} I_l \hat{c}_{m-i-l},\ m \ge 2,
\end{aligned}$$

where $\hat{a}_i,\ \hat{b}_i,\ \hat{c}_i,\ i \ge 1$, are given by (3.19). After interchanging the summing in the above equality, i.e.

$$\sum_{i=1}^{m-1} \sum_{k=0}^{i-1} = \sum_{k=0}^{m-2} \sum_{i=k+1}^{m-1},\ \sum_{i=1}^{m-1} \sum_{l=0}^{m-i-1} = \sum_{l=0}^{m-2} \sum_{i=1}^{m-1-l},\ \sum_{i=1}^{m-1} \sum_{k=0}^{i-1} \sum_{l=0}^{m-i-1} = \sum_{k=0}^{m-2} \sum_{l=0}^{(m-2)-k} \sum_{i=k+1}^{m-(l+1)},$$

we may arrive at

$$
\begin{aligned}
-2I_m &= \sum_{i=1}^{m-1} I_i I_{m-i} + \sum_{k=0}^{m-2} \sum_{l=0}^{(m-2)-k} \sum_{i=k+1}^{m-(l+1)} I_k I_l (\hat{a}_{i-k}\hat{a}_{m-i-l} + \hat{b}_{i-k}\hat{c}_{m-i-l}) \\
&\quad - \sum_{k=0}^{m-2} \sum_{i=k+1}^{m-1} I_k I_{m-i}\hat{a}_{i-k} - \sum_{l=0}^{m-2} \sum_{i=1}^{m-l-1} I_l I_i \hat{a}_{m-i-l} - 2\sum_{i=0}^{m-1} I_i \hat{a}_{m-i} \\
&:= B_1 + B_2 + B_3 + B_4 + B_5,\ m \geq 2. \qquad (4.8)
\end{aligned}
$$

Further we have

$$
\begin{aligned}
B_3 &= -\sum_{k=0}^{m-2} (\sum_{i=k+2}^{m} + \sum_{i=k+1} - \sum_{i=m}) I_k I_{m-i}\hat{a}_{i-k} \\
&= -\sum_{k=0}^{m-2} \sum_{i=k+2}^{m} I_k I_{m-i}\hat{a}_{i-k} - \sum_{k=0}^{m-2} I_k I_{m-k-1}\hat{a}_1 + \sum_{k=0}^{m-2} I_k \hat{a}_{m-k} \\
&= -\sum_{k=0}^{m-2} \sum_{l=0}^{(m-2)-k} I_k I_l \hat{a}_{m-(k+l)} - \sum_{k=0}^{m-2} I_k I_{m-k-1}\hat{a}_1 + \sum_{k=0}^{m-2} I_k \hat{a}_{m-k},
\end{aligned}
$$

$$
\begin{aligned}
B_4 &= -\sum_{l=0}^{m-2} (\sum_{i=0}^{m-l-2} - \sum_{i=0} + \sum_{i=m-l-1}) I_l I_i \hat{a}_{m-i-l} \\
&= -\sum_{k=0}^{m-2} \sum_{l=0}^{(m-2)-k} I_k I_l \hat{a}_{m-(k+l)} + \sum_{l=0}^{m-2} I_l \hat{a}_{m-l} - \sum_{l=0}^{m-2} I_l I_{m-l-1}\hat{a}_1.
\end{aligned}
$$

Therefore the latter three terms in the right hand side of (4.8) becomes

$$
B_3 + B_4 + B_5 = -2\sum_{k=0}^{m-2} \sum_{l=0}^{(m-2)-k} I_k I_l \hat{a}_{m-(k+l)} + \frac{1}{4} \sum_{\substack{k+l=m-1 \\ k,l\geq 0}} I_k I_l F_1.
$$

In this way, from (4.8) we obtain

$$
\begin{aligned}
I_m &= -\frac{1}{2}\sum_{i=1}^{m-1} I_i I_{m-i} - \frac{1}{8}\sum_{k=0}^{m-2} \sum_{l=0}^{(m-2)-k} I_k I_l F_{m-(k+l)} - \frac{1}{8} \sum_{\substack{k+l=m-1 \\ k,l\geq 0}} I_k I_l F_1 \\
&= -\frac{1}{2}\sum_{i=1}^{m-1} I_i I_{m-i} - \frac{1}{8} \sum_{\substack{k+l\leq m-1 \\ k,l\geq 0}} I_k I_l F_{m-(k+l)},\ m \geq 2, \qquad (4.9)
\end{aligned}
$$

by which we can determine any I_m, $m \geq 2$, starting with $I_1 = -\frac{1}{8}F_1$. It is not difficult to find a homogeneous property among the terms of (4.9). Thus we may assume that

$$I_m = \sum_{n=1}^{m} d_n \sum_{\substack{i_1+\cdots+i_n=m \\ i_1,\cdots,i_n\geq 1}} F_{i_1}\cdots F_{i_n}, \ m \geq 2.$$

In general, the coefficients d_n, $1 \leq n \leq m$, should depend on m. But the following deduction implies that this assumption is possible. First from (4.9) we easily have $I_2 = \frac{3}{128}F_1^2 - \frac{1}{8}F_2$, which leads to $d_2 = \frac{3}{128}$. When $m \geq 3$, (4.9) becomes

$$I_m = -\frac{1}{2}\sum_{i=1}^{m-1} I_i I_{m-i} - \frac{1}{8}F_m - \frac{1}{4}\sum_{k=1}^{m-1} I_k F_{m-k} - \frac{1}{8}\sum_{\substack{k+l\leq m-1 \\ k,l\geq 1}} I_k I_l F_{m-(k+l)}, \qquad (4.10)$$

in which the coefficients of the F_1^m yields the recursion relation (4.6). In what follows, we want to prove that I_m, $m \geq 0$, determined above satisfy the relation (4.10), indeed. This may be shown by combining the following three equalities. First we have

$$\begin{aligned}
\sum_{i=1}^{m-1} I_i I_{m-i} &= \sum_{i=1}^{m-1}\sum_{k=1}^{i} d_k \sum_{\substack{i_1+\cdots+i_k=i \\ i_1,\cdots,i_k\geq 1}} F_{i_1}\cdots F_{i_k} \sum_{l=1}^{m-i} d_l \sum_{\substack{j_1+\cdots+j_l=m-i \\ j_1,\cdots,j_l\geq 1}} F_{j_1}\cdots F_{j_l} \\
&= \sum_{k=1}^{m-1}\sum_{i=k}^{m-1}\sum_{l=1}^{m-i} d_k d_l \sum_{\substack{i_1+\cdots+i_k=i \\ i_1,\cdots,i_k\geq 1}} F_{i_1}\cdots F_{i_k} \sum_{\substack{j_1+\cdots+j_l=m-i \\ j_1,\cdots,j_l\geq 1}} F_{j_1}\cdots F_{j_l} \\
&= \sum_{k=1}^{m-1}\sum_{l=1}^{m-k}\sum_{i=k}^{m-l} d_k d_l \sum_{\substack{i_1+\cdots+i_k=i \\ i_1,\cdots,i_k\geq 1}} F_{i_1}\cdots F_{i_k} \sum_{\substack{j_1+\cdots+j_l=m-i \\ j_1,\cdots,j_l\geq 1}} F_{j_1}\cdots F_{j_l} \\
&= \sum_{n=2}^{m} \sum_{\substack{k+l=n \\ k,l\geq 1}} d_k d_l \sum_{\substack{i_1+\cdots+i_n=m \\ i_1,\cdots,i_n\geq 1}} F_{i_1}\cdots F_{i_n}.
\end{aligned}$$

Similarly we can get the other two equalities

$$\sum_{k=1}^{m-1} I_k F_{m-k} = \sum_{n=2}^{m} d_{n-1} \sum_{\substack{i_1+\cdots+i_n=m \\ i_1,\cdots,i_n\geq 1}} F_{i_1}\cdots F_{i_n},$$

$$\sum_{\substack{k+l\leq m-1 \\ k,l\geq 1}} I_k I_l F_{m-(k+l)} = \sum_{n=2}^{m-1} \sum_{\substack{i+j=n \\ i,j\geq 1}} d_i d_j \sum_{\substack{i_1+\cdots+i_{n+1}=m \\ i_1,\cdots,i_{n+1}\geq 1}} F_{i_1}\cdots F_{i_{n+1}}.$$

Therefore the proof is finished. ▮

In the proof of the above theorem, $\tilde{a}_i,\ \tilde{b}_i,\ \tilde{c}_i,\ i\ge 0$, are the expressions of $a_i,\ b_i,\ c_i,\ i\ge 0$, corresponding to non-zero boundary condition on the potential u under which the equality $\Psi(\delta\lambda/\delta u)=\lambda(\delta\lambda/\delta u)$ doesn't hold. Note that under zero boundary condition, we have $\Psi(\delta\lambda/\delta u)=\lambda(\delta\lambda/\delta u)$ and thus the expressions of $a_i,\ b_i,\ c_i,\ i\ge 0$, are just the functions $\hat{a}_i,\ \hat{b}_i,\ \hat{c}_i,\ i\ge 0$.

Theorem 4.2 *If P_i and $Q_i,\ 1\le i\le 3$, solve the nonlinearized spatial system (3.8), then there exist N integrals of motion $\alpha_i,\ 0\le i\le N-1$, of (3.8) such that*

$$q=\sqrt{2}(<P_1,BQ_2>+<P_2,BQ_3>),\ r=\sqrt{2}(<P_2,BQ_1>+<P_3,BQ_2>)$$

solve the following N-th order stationary AKNS equation

$$K_N+\sum_{i=0}^{N-1}\alpha_iK_i=0,$$

where $K_i,\ 0\le i\le N$, are defined by (2.10).

Proof: Noting that the expressions (4.3) and (4.4) of $\widetilde{b}_{i+1},\ \widetilde{c}_{i+1},\ i\ge 0$, we can compute that

$$\begin{aligned}
&\sum_{i=0}^N\alpha_i\widetilde{K}_i=\sum_{i=0}^N\alpha_iJ\begin{pmatrix}\widetilde{c}_{i+1}\\ \widetilde{b}_{i+1}\end{pmatrix}\\
&=J\sum_{i=0}^N\alpha_i\sum_{j=0}^iI_j\begin{pmatrix}\sqrt{2}(<A^{i-j}P_2,BQ_1>+<A^{i-j}P_3,BQ_2>)\\ \sqrt{2}(<A^{i-j}P_1,BQ_2>+<A^{i-j}P_2,BQ_3>)\end{pmatrix}\\
&=\sqrt{2}J\sum_{i=0}^N\alpha_i\sum_{j=0}^iI_j\sum_{k=1}^N\lambda_k^{i-j}\mu_k\begin{pmatrix}\phi_{2k}\psi_{1k}+\phi_{3k}\psi_{2k}\\ \phi_{1k}\psi_{2k}+\phi_{2k}\psi_{3k}\end{pmatrix}\\
&=\sqrt{2}J\sum_{k=1}^N\mu_k\Big(\sum_{i=0}^N\sum_{j=0}^i\alpha_iI_j\lambda_k^{i-j}\Big)\begin{pmatrix}\phi_{2k}\psi_{1k}+\phi_{3k}\psi_{2k}\\ \phi_{1k}\psi_{2k}+\phi_{2k}\psi_{3k}\end{pmatrix}\\
&=\sqrt{2}J\sum_{k=1}^N\mu_k\sum_{l=0}^N\Big(\sum_{i=l}^N\alpha_iI_{i-l}\Big)\lambda_k^l\begin{pmatrix}\phi_{2k}\psi_{1k}+\phi_{3k}\psi_{2k}\\ \phi_{1k}\psi_{2k}+\phi_{2k}\psi_{3k}\end{pmatrix}.
\end{aligned}$$

Secondly we set

$$G(\lambda)=\prod_{i=1}^N(\lambda-\lambda_i)=\sum_{i=0}^N\beta_i\lambda^i=\lambda^N+\sum_{i=0}^{N-1}\beta_i\lambda^i.$$

Let us now choose

$$\sum_{i=l}^N\alpha_iI_{i-l}=\beta_l,\ 0\le l\le N,$$

which determines recursively

$$\alpha_N=\beta_N=1,\ \alpha_l=\beta_l-\sum_{i=l+1}^{N}\alpha_iI_{i-l},\ 0\le l\le N-1,$$

due to $I_0=1$. The $\alpha_i,\ 0\le i\le N-1$, are all integrals of motion of (3.8) since they are functions of $I_i,\ 0\le i\le N$. Further by $G(\lambda_k)=0,\ 1\le k\le N$, we see that $\sum_{k=0}^N\alpha_i\widetilde{K}_i=0$, which completes the proof. ▮

The above theorem also implies that the potential determined by the Bargmann symmetry constraint (3.5) is a finite gap potential of the spectral problem (2.1). It is worth saying that the coefficients $\alpha_i,\ 1\le i\le N-1$, in the stationary equations are generally not constants (just integrals of motion).

Theorem 4.3 *Under the control of the nonlinearized spatial system (3.8), the nonlinearized temporal systems (3.9) for $n\ge 0$ can also be rewritten as the Hamiltonian systems*

$$P_{it_n}=\{P_i,H_n\}=-B^{-1}\frac{\partial H_n}{\partial Q_i},\ Q_{it_n}=\{Q_i,H_n\}=B^{-1}\frac{\partial H_n}{\partial P_i},\quad i=1,2,3 \tag{4.11}$$

with the Hamiltonian functions

$$H_n=-\frac{1}{4}\sum_{m=0}^{n}\frac{d_m}{m+1}\sum_{\substack{i_1+\cdots+i_{m+1}=n+1\\ i_1,\cdots,i_{m+1}\ge 1}}F_{i_1}\cdots F_{i_{m+1}}\,,\ n\ge 0,$$

where $d_0=1$ and F_m, $m\ge 1$, are defined by (3.18).

Proof: We only prove the former equality of (4.11). We know that under the control of the nonlinearized spatial system (3.8), the results in Theorem 4.1 holds. Hence we have

$$\begin{aligned}
P_{1t_n}&=2\sum_{i=0}^{n}\widetilde{a}_iA^{n-i}P_1+\sqrt{2}\sum_{i=0}^{n}\widetilde{b}_iA^{n-i}P_2\\
&=-2A^nP_1+2\sum_{i=1}^{n}(\sum_{k=0}^{i-1}I_k\hat{a}_{i-k}-I_i)A^{n-i}P_1+\sqrt{2}\sum_{i=1}^{n}\sum_{k=0}^{i-1}I_k\hat{b}_{i-k}A^{n-i}P_2\\
&=-2\sum_{k=0}^{n}I_kA^{n-k}P_1+\sum_{k=0}^{n-1}I_k\sum_{i=k+1}^{n}2\hat{a}_{i-k}A^{n-i}P_1+\sum_{k=0}^{n-1}I_k\sum_{i=k+1}^{n}\sqrt{2}\hat{b}_{i-k}A^{n-i}P_2\\
&=\frac{1}{4}I_nB^{-1}\frac{\partial F_1}{\partial Q_1}+\frac{1}{4}\sum_{k=0}^{n-1}I_kB^{-1}\frac{\partial F_{n-k+1}}{\partial Q_1}=\frac{1}{4}\sum_{k=0}^{n}I_kB^{-1}\frac{\partial F_{n-k+1}}{\partial Q_1},
\end{aligned}$$

where $\hat a_i,\ \hat b_i,\ i\ge 1$, are given by (3.19). We further note the expression of $I_m,\ m\ge 0$, defined by (4.5) and then we may make the following performance

$$\begin{aligned}
P_{1t_n} &= \frac14 I_0B^{-1}\frac{\partial F_{n+1}}{\partial Q_1}+\frac14\sum_{k=1}^{n}\sum_{m=1}^{k} d_m \sum_{\substack{i_1+\cdots+i_m=k\\ i_1,\cdots,i_m\ge 1}} F_{i_1}\cdots F_{i_m}B^{-1}\frac{\partial F_{n-k+1}}{\partial Q_1}\\
&= \frac14 I_0B^{-1}\frac{\partial F_{n+1}}{\partial Q_1}+\frac14\sum_{m=1}^{n} d_m \sum_{k=m}^{n}\sum_{\substack{i_1+\cdots+i_m=k\\ i_1,\cdots,i_m\ge 1}} F_{i_1}\cdots F_{i_m}B^{-1}\frac{\partial F_{n-k+1}}{\partial Q_1}\\
&= \frac14 I_0B^{-1}\frac{\partial F_{n+1}}{\partial Q_1}+\frac14\sum_{m=1}^{n} \frac{d_m}{m+1}B^{-1}\frac{\partial}{\partial Q_1}\sum_{\substack{i_1+\cdots+i_{m+1}=n+1\\ i_1,\cdots,i_{m+1}\ge 1}} F_{i_1}\cdots F_{i_{m+1}}\\
&= \frac14 B^{-1}\frac{\partial}{\partial Q_1}\sum_{m=0}^{n} \frac{d_m}{m+1}\sum_{\substack{i_1+\cdots+i_{m+1}=n+1\\ i_1,\cdots,i_{m+1}\ge 1}} F_{i_1}\cdots F_{i_{m+1}} = -B^{-1}\frac{\partial H_n}{\partial Q_1},
\end{aligned}$$

where we have accepted $d_0=1$. The above manipulation is fulfilled for the case of $n\ge 1$. The case of $n=1$ needs only a simple calculation. Thus the former equality of (4.11) is true for $n\ge 0$. The latter equality of (4.11) may be proved similarly. The proof is completed. ▮

The above theorem permits us to establish a sort of involutive solutions to AKNS soliton equations, which exhibits a kind of separation of variables for AKNS soliton equations. This is the following result.

Theorem 4.4 *The n-th AKNS soliton equation $u_{t_n}=K_n$ has the involutive solution with separated variables x, t_n*

$$\left\{\begin{aligned}
q &= \sqrt2(< g_H^x g_{H_n}^{t_n}P_1(0,0), Bg_H^x g_{H_n}^{t_n}Q_2(0,0)> \\
&\qquad + < g_H^x g_{H_n}^{t_n}P_2(0,0), Bg_H^x g_{H_n}^{t_n}Q_3(0,0)>),\\
r &= \sqrt2(< g_H^x g_{H_n}^{t_n}P_2(0,0), Bg_H^x g_{H_n}^{t_n}Q_1(0,0)> \\
&\qquad + < g_H^x g_{H_n}^{t_n}P_3(0,0), Bg_H^x g_{H_n}^{t_n}Q_2(0,0)>).
\end{aligned}\right. \tag{4.12}$$

where g_G^y denotes the Hamiltonian phase flow of G with a variable y and $P_i(0,0)$ and $Q_i(0,0),\ 1\le i\le 3$, may be arbitrary initial value vectors.

Proof: Let

$$P_i(x,t_n)=g_H^x g_{H_n}^{t_n}P_i(0,0),\ Q_i(x,t_n)=g_H^x g_{H_n}^{t_n}Q_i(0,0),\ 1\le i\le 3.$$

Then $P_i(x,t_n)$ and $Q_i(x,t_n),\ 1\le i\le 3$, solve the nonlinearized spatial system (3.8) and the Hamiltonian system (4.11). However under the control of (3.8), (4.11) is equivalent to the nonlinearized temporal system (3.9). This shows that $P_i(x,t_n)$ and $Q_i(x,t_n)$ also solve (3.8) and (3.9), simultaneously. Therefore the compatibility condition of (3.8) and (3.9) is satisfied, i.e. (4.12) determines a solution to $u_{t_n}=K_n$. In addition, since $\{H,H_n\}=0$, the Hamiltonian phase flows g_H^x, $g_{H_n}^{t_n}$ may commute with each other. It follows that the resulting solution (4.12) is involutive. The proof is finished. ▮

5 Conclusions and remarks

We have introduced a three-by-three matrix spectral problem for the usual AKNS soliton hierarchy and proposed the corresponding Bargmann symmetry constraint on this AKNS hierarchy. Moreover we have exhibited an explicit Poisson algebra

$$\{\bar{F}_j,\ 1\le j\le N,\ F_m,\ m\ge 1\} \tag{5.1}$$

on the symplectic manifold $(\mathbb{R}^{6N},\omega^2)$ and further a binary nonlinearization procedure is manipulated along with a sort of involutive solutions to AKNS soliton equations. We have also proved rigorously that this Poisson algebra suffices for proving complete integrability of the nonlinearized Lax systems and thus the nonlinearized spatial system (3.8) and the nonlinearized temporal systems (3.10) are all integrable in the Liouville sense, indeed. Lax representations for the nonlinearized Lax systems are obtained as a product.

The above Poisson algebra is different from one given in Ref. [3]. It possesses a combined matrix B coming from a general symmetry constraint and the resulting integrable systems has non-standard canonical structures. To the authors' knowledge. this kind of results is introduced for the first time.

It is well known that there are a lot of results about nonlinearization of scalar spectral problems or 2×2 matrix spectral problems but there are few results in the case of 3×3 matrix spectral problems. In this paper, we focus our discussion mainly on binary nonlinearization on a 3×3 matrix spectral problem and the integrability of the resulting nonlinearized Lax systems. We have successfully extended the binary nonlinearization theory to that 3×3 matrix spectral problem for AKNS hierarchy. Furthermore based upon the obtained Lax representations, we may also construct r-matrices and canonically conjugate variables and discuss separation of variables for the nonlinearized Lax systems, which will be left to a future publication.

It should also be pointed out that the Neumann symmetry constraint and the higher order symmetry constraints

$$K_{-1}=J\sum_{j=1}^N E_j\frac{\delta\lambda_j}{\delta u},\ K_m=JG_m=J\sum_{j=1}^N E_j\frac{\delta\lambda_j}{\delta u},\ (m\ge 1), \tag{5.2}$$

may be considered. These sorts of symmetry constraints are somewhat different from the Bargmann symmetry constraints because K_{-1} is a constant vector and the conserved covariants G_m, $m \geq 1$, involve the differential of the potential u with respect to the space variable x. In order to discuss them, we are required to introduce a new symplectic submanifold of the Euclidean spaces in the case of the Neumann constraint and new dependent variables, i.e. the so-called Jacobi-Ostrogradsky coordinates [11], in the case of higher order constraints. Similarly, we can consider the corresponding τ-symmetry (time first order dependent symmetry) constraints or more generally, time polynomial dependent symmetry constraints. Note that the similar Bargmann symmetry constraints have also been carefully analyzed for KP hierarchy [31] and the symmetries in the right hand side of the Bargmann symmetry constraints may be taken as sources of soliton equations [32].

We remark that the finite dimensional Hamiltonian systems generated by nonlinearization technique depend on the starting spectral problems. Therefore the same soliton equation may relate to different finite dimensional Hamiltonian systems once it possesses different Lax representations. AKNS soliton equations are exactly such examples. But we don't know if there exists an interrelation among the different finite dimensional Hamiltonian systems generated from the same soliton equation. In the binary nonlinearization procedure itself, there also exist some intriguing open problems. For example, why do the nonlinearized spatial system and the nonlinearized temporal systems for $n \geq 0$ under the control of the nonlinearized spatial system always possess Hamiltonian structures? We don't know either whether or not the nonlinearized temporal systems for $n \geq 0$ are themselves integrable soliton equations without the control of the nonlinearized spatial system. These problems are worth studying in order to enrich integrable structures of soliton equations.

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