SYMPLECTIC STRUCTURES, THEIR BÄCKLUND TRANSFORMATIONS AND HEREDITARY SYMMETRIES

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It is shown that compatible symplectic structures lead in a natural way to hereditary symmetries. (We recall that a hereditary symmetry is an operator-valued function which immediately yields a hierarchy of evolution equations, each having infinitely many commuting symmetries all generated by this hereditary symmetry. Furthermore this hereditary symmetry usually describes completely the soliton structure and the conservation laws of these equations). This result then provide us with a method for constructing hereditary symmetries and hence exactly solvable evolution equations.

In addition we show how symplectic structures transform under Bäcklund transformations. This leads to a method for generating a whole class of symplectic structures from a given one.

Several examples and applications are given illustrating the above results. Also the connection of our results with those of Gelfand and Dikii, and of Magri is briefly pointed out.

1. Introduction and basic notions

It is well known that certain evolution equations, like the Korteweg-deVries equation, possess various surprising features such as infinitely many symmetries and conserved quantities, N-soliton solutions, Bäcklund transformations, etc. Recently there has also been progress in understanding the Hamiltonian [12, 3, 20] and bi-Hamiltonian [18] structure of these, so called, exactly solvable evolution equations. An evolution equation is said to be a Hamiltonian system if it can be written in the form $u_t = \theta(u)f(u)$, where $\theta(u)$ is an implectic operator (i.e. roughly speaking $\theta^{-1}(u)$ is symplectic) and where $f(u)$ is a gradient function (i.e. $f(u)$ is the gradient of a suitable potential).

An equation is a bi-Hamiltonian system if $u_t = \theta_1(u)f_1(u) = \theta_2(u)f_2(u)$.

In this paper we study evolution equations in the form

$$u_t = K(u), \quad u \in M,$$

where $u$ is in some manifold $M$, and $K$ is a suitable $C^\omega$ vector field. Clearly vector equations as well as integral equations, such as the Benjamin-Ono equation [3, 21], are included in (1).

Basic role in our approach is played by an operator $\Phi(u)$ which maps symmetries§ of (1) onto symmetries of (1). Such an operator is called a strong symmetry of (1) [10] (or a recursion operator [19]) and has the following remarkable properties: i) its eigenfunctions are also symmetries, which actually characterize the N-soliton solutions [10]; ii) its transpose $\Phi^*(u)$ maps gradients of conserved quantities of (1) onto gradients of conserved quantities of (1); iii)

§Throughout this paper by symmetries we really mean infinitesimal generators of symmetries [6].
the eigenfunctions of its transpose $\Phi^*(u)$ are also gradients of conserved quantities. (In the case of the KdV, $\Phi^*$ is the squared-eigenfunction operator [1]. This is because the squares of the eigenfunctions of the Schrödinger operator are the gradients of the eigenvalues of the Schrödinger operator [11], i.e. there are gradients of conserved quantities.)

For the exact solvability of an equation it is crucial that its infinitely many symmetries commute (or equivalently, that its conserved quantities are in involution). This is the case if the operator $\Phi(u)$ has a certain property called the hereditary property (its mathematical definition is given in section 3). An operator with this property was called by Fuchssteiner a hereditary symmetry [10]. It also follows from the definitions of strong and hereditary symmetries that, if an operator is hereditary then it is immediately a strong symmetry for any of the equations $u_t = \Phi(u)^* u_n$, $n = 1, 2, 3, \ldots$.

From the last statement in follows that it is very desirable to have methods for constructing hereditary symmetries, since these hereditary symmetries immediately yield hierarchies of evolution equations each possessing infinitely many commuting symmetries. One such method was given in [7], where it is shown that Bäcklund transformations yield transformations between hereditary symmetries, thus they generate a class of hereditary symmetries from a given one.

In this paper we show that

a) If $\theta_1$, $\theta_2$ are implictic operators which are compatible (i.e. $\theta_1 + \theta_2$ is also implictic) then the operator $\Phi = \theta_1 \theta_1^*$ is hereditary.

b) If $B(u, s) = 0$ is a Bäcklund transformation and $\theta(u)$ is an implictic operator then $\Omega(s) = T\theta(u)T^*$, $T = B_s^{-1}B_u$ is also implictic.

By using various Miura type Bäcklund transformations and the above results we have constructed many new hereditary symmetries and hence many new exactly solvable evolution equations. Some of them are given here, others will appear in future publications.

Some of the results presented here have been announced in [8].

This paper is organized as follows: We first introduce some notation and the basic notions used throughout our paper (see below), and then in section 2 we review symplectic and implictic structures as well as the bi-Hamiltonian formulation of evolution equations [18]. Sections 3 and 5 contain the main results of this paper (see (a) and (b) above). Several examples and applications are given in section 6. Because our notation and basic notions are not well known, we have included section 4 where we motivate and illustrate all notions used in this paper. We hope that this section will be particularly valuable to the reader familiar with soliton theory, but unfamiliar with the notions used here, because it will help him to establish a correspondence with more familiar notions.

**Notation**

We assume that the typical fibre of the tangent bundle of $M$ is the real-linear space $S$ and let $S^*$ be its dual space. For $a \in S^*$ and $u \in S$, $\langle a, u \rangle$ denotes the application of the linear functional given by $a$ to $u$. The transpose of an operator $T: S \rightarrow S$ with respect to this duality is denoted by $T^*$. Often we consider functions $\phi$ on $S$ attaining values either in $S^*$ or in the space of operators $S^* \rightarrow S$ (or $S \rightarrow S$). We only deal with differentiable functions, i.e. the directional derivative in $S$,

$$\phi'(u)[v] = \frac{\partial}{\partial \epsilon} \phi(u + \epsilon v)\bigg|_{\epsilon = 0},$$

has to exist and the map $\phi'(u)[\cdot]$ is assumed to be linear. In addition we always assume that the chain rule holds (in a topological context that means that we deal with Hadamard derivatives). If no confusion can arise we write $\phi$ or $\phi'[\cdot]$ instead of $\phi(u)$ and $\phi'(u)[\cdot]$.

An operator $\theta: S^* \rightarrow S$ is called symmetric if
(a, b) = (b, a) for all \( a, b \in S^* \), and skew-symmetric if \( (a, b) = -(b, a) \) always holds. For operators \( J: S \rightarrow S^* \) these notions are defined analogously. A function \( \phi: S \rightarrow S^* \) is said to be a gradient function if it has a potential, i.e. a map \( p: S \rightarrow \mathbb{R} \) such that

\[
(\phi(u), v) = p'(u)[v], \quad \text{for all } u, v \in S.
\]

A necessary and sufficient condition for the existence of a potential is that \( \phi'(v) \) is always symmetric. Then a potential \( p \) is given by

\[
p(v) = \int_0^1 (\phi(\lambda v), v) \, d\lambda . \tag{2}
\]

**Basic notions**

This paper is concerned with evolution equations (1). An important role with respect to symmetries is then played by the derivative \( K' \triangleq K'(u)[\cdot] \) and its transposed \( K'^* \).

**Definition 1.**

(i) A function \( \sigma: M \rightarrow S \) is called a symmetry of (1) if we have always

\[
\sigma'[K] - K'[\sigma] = 0.
\]

(ii) \( \gamma: M \rightarrow S^* \) is said to be a conserved covariant of (1) if we have always

\[
\gamma'[K] + K'^*[\gamma] = 0 .
\]

The reason for this name is that if \( \gamma \) also satisfies \( \gamma' = (\gamma')^* \), then \( \gamma \) is the gradient of a conserved quantity (see section 4).

(iii) A function \( \Phi \) from \( M \) into the space of operators \( S \rightarrow S \) is called a strong symmetry of (1) if we have always

\[
\Phi'[K] - [K', \Phi] = 0.
\]

(iv) A function \( \theta \) from \( M \) into the space of operators \( S^* \rightarrow S \) is said to be a Noether operator for (1) if we have always

\[
\theta'[K] - \theta K'^* - K'\theta = 0 .
\]

(v) A function \( J \) from \( M \) into the space of operators \( S \rightarrow S^* \) is called an inverse Noether operator if we have always:

\[
J'[K] + JK' + K'^*J = 0 .
\]

Note that all the functions considered above are functions of the variable \( u \in M \), and by 'always' we mean that the relations above hold identically in this variable.

**Remark 1.**

All the notions we have considered above are intimately connected with the evolution given by (1). To see this, take an arbitrary solution \( u(t) \) of (1), put \( K'(t) = K'(u(t)) \), \( K'^*(t) = K'^*(u(t)) \) and consider the corresponding perturbation equation

\[
v_t = K'(t)v, \quad v(t) \in S \tag{3}
\]

and its adjoint

\[
w_t = -K'^*(t)w, \quad w(t) \in S^*. \tag{4}
\]

Then for \( \sigma(t) = \sigma(u(t)) \), \( \gamma(t) = \gamma(u(t)) \), etc., we have:

(i) \( \sigma(t) \) is a solution of (3),
(ii) \( \gamma(t) \) is a solution of (4),
(iii) \( \Phi(t) \) maps solutions of (3) into solutions of (3),
(iv) \( \Phi(t)^* \) maps solutions of (4) into solutions of (4),
(v) \( \theta(t) \) maps solutions of (4) into solutions of (3),
(vi) \( J(t) \) maps solutions of (3) into solutions of (4).

Furthermore—in case that the initial value prob-
lem for (3)-(4) is properly posed—the validity of (i) to (v) (for any solution \( u(t) \) of (1)) is equivalent to the properties required in definition 1.

This remark suggests that there are very many relations between these notions. Let us list the basic ones:

Remark 2.
(i) \( \Phi \sigma \) and \( \theta \gamma \) are symmetries, \( J\sigma \) is a conserved covariant;
(ii) \( \theta J \) is a strong symmetry;
(iii) \( \Phi \theta \) is a Noether operator and \( J\Phi \) is an inverse Noether operator;
(iv) taking the inverse (if it exists) goes from Noether operators to inverse Noether operators (and vice versa);
(v) the set of strong symmetries is an algebra.

All these statements can be checked by elementary differential calculus.

2. Symplectic-implectic operators, Hamiltonian systems and Magri’s results

An operator-valued function \( J(u): S \to S^* \), \( u \in M \), which is skew-symmetric is said to be a symplectic operator if the bracket defined on \( S^3 \) by

\[
[a, b, c] = \langle J'(u)[a]b, c \rangle 
\]  

satisfies the Jacobi identity

\[
[a, b, c] + [b, c, a] + [c, a, b] = 0 \quad \text{for all } a, b, c \in S.
\]  

(The above means that the exterior derivative of the corresponding two-form vanishes, i.e. the two-form is closed.)

If \( J \) has an inverse \( \theta = J^{-1} \) there is a corresponding identity for \( \theta \). In that case the Jacobi identity is fulfilled for the bracket \( \{ \} \) defined on \( S^{*3} \) by

\[
\{a, b, c\} = \langle a, b, c\rangle(u) = \langle b, \theta'(u)[\theta(u)a]c \rangle .
\]  

However, the bracket \( \{ \} \) makes also sense (and plays an important role) even for operators \( \theta \) which are not invertible.

Definition 2.
An operator-valued function \( \theta(u): S^* \to S \), \( u \in M \), which is skew-symmetric is called implectic if the bracket defined by (6) satisfies the Jacobi identity, i.e.,

\[
\langle b, \theta'[^\theta a]c \rangle + \langle c, \theta'[^\theta b]a \rangle + \langle a, \theta'[^\theta c]b \rangle = 0, \quad \text{for all } a, b, c \in S^*. 
\]  

Implectic is—by abuse of language—a short form for inverse symplectic; in the literature there are also other names for this structure, c.f. [2] or [17]. We do not call \( \theta \) co-symplectic because this would imply that \( \theta^{-1} \) exists; however, the existence of \( \theta^{-1} \) is not necessary here.

It should be observed that every constant skew-symmetric operator is implectic. Obviously the inverse of symplectic operators (if they exist) are implectic.

We shall now give a formula playing an important role in the characterization of implectic operators: Let \( f \) be a function \( S \to S^* \) and let \( \theta(u): S^* \to S, \ u \in M \), be a skew-symmetric operator. Then if

\[
G(u) = \theta(u)f(u) 
\]  

it follows that for all \( a, b \in S^* \)

\[
\langle b, (\theta'[G] - \theta G'^* - G'\theta)a \rangle 
\]

\[
= \langle b, \theta'[f]a \rangle + \langle f, \theta'[^\theta a]b \rangle
\]

\[
+ \langle a, \theta'[^\theta b]f \rangle - \langle b, \theta'(f' - f'^*)a \rangle .
\]  

The above formula is a consequence of the skew-symmetry of \( \theta \) and of the obvious iden-
\( G'[\cdot] = \theta'[\cdot] f + \theta f'[\cdot] \).
\(<b, \theta G^*[a]> = -\langle \theta b, G^*[a] > = -\langle a, G'[\theta b] > . \)

**Proposition 1.** Let the \( \theta(u), u \in M \), be skew-symmetric operators \( S^* \rightarrow S \). Then the following are equivalent:

(i) \( \theta \) is implectic;

(ii) \( \theta \) is a Noether operator for every evolution equation of the form

\( u_t = \theta(u)f(u), \ f \text{ gradient function}; \quad (9) \)

(iii) for all gradient functions \( f \) and \( g \) we have

\(<(\theta f)'[\theta g] - (\theta g)'[\theta f] = \theta\langle f, \theta g \rangle'[\cdot] . \)

**Proof.**

(i)\( \Rightarrow \)(ii): Put \( G(u) = \theta(u)f(u) \). Then since \( f \) is a gradient function we have \( f' = f^{**} \). Hence the right side of (8b) vanishes by virtue of (i). So, the left side must be equal to zero, and \( \theta \) is a Noether operator for the evolution equation \( u_t = G(u) \).

(ii)\( \Rightarrow \)(i): One has to keep in mind that constant functions are gradient functions. So, from (ii) it follows that the right side of (8b) must vanish for arbitrary constant \( f \). This implies (i).

(i)\( \Leftrightarrow \)(iii): Performing the differentiations in (iii) for constant functions \( f, g \) one sees that (ii) is a special case of (iii). For nonconstant gradient functions \( f \) and \( g \) the contributions due to the \( f' \) and \( g' \) of the right and left side of (iii) cancel out.

Let us list some corollaries of eq. (8b) and of the above proposition.

**Remark 3.**

(i) (Gelfand and Dikii or Magri) If \( f \) and \( g \) are gradients and \( \theta \) is implectic one usually defines a Poisson bracket by

\([f, g] = \langle f, \theta g > \).

Further in the set of functions \( \sigma; S \rightarrow S \) one considers the Lie-algebra commutator

\([\sigma_1, \sigma_2] = \sigma_1'[\sigma_2] - \sigma_2'[\sigma_1] . \)

Then (iii) of prop. 1 says that the operation \( \theta(u) \times \text{gradient maps the Poisson bracket onto the commutator of the Lie algebra. This is well known (cf. [13] or [18])}. \)

(ii) (Noether's law). Usually the equivalence of (i)\( \Leftrightarrow \)(iii) of prop. 1 is used to show that \( \theta \) maps the gradients of conservation laws of the Hamiltonian system (9) onto symmetries of that system (cf. [18]).

But (i)\( \Leftrightarrow \)(ii) actually tells us more. Namely, that \( \theta \) maps conserved covariants of (9) onto symmetries (whether the conserved covariants are gradients or not).

(iii) Assume that \( \theta \) is invertible or that \( \theta(u)S^* \) is weak-star dense in \( S \). Then if \( \theta \) is a Noether operator for the equation \( u_t = \theta(u)f(u) \) and \( \theta \) is implectic, then \( f \) must be a gradient function. This follows directly from (8b):

\(<a, \theta(f' - f^{**})\theta b > = 0, \quad \text{for all } a, b \in S^* . \)

Hence, \( f' \) must be symmetric.

Let us recall that a Hamiltonian system is an evolution equation of the form

\( u_t = \theta(u)f(u), \ u(t) \in M \),

where \( \theta \) is implectic and \( f \) is a gradient function. We call a system bi-Hamiltonian if it can be written in two different ways as a Hamiltonian system, i.e.

\( u_t = K(u) \quad (10a) \)

where

\( K(u) = \theta_1(u)f_1(u) = \theta_2(u)f_2(u) , \quad (10b) \)

with implectic \( \theta_1, \theta_2 \) and gradient functions \( f_1, f_2 \).
Now, using (iii), (iv) of remark 2 and prop. 1 (ii) we immediately obtain,

Magri's construction of a strong symmetry: Consider the bi-Hamiltonian system (10) and assume that \( \theta_1 \) is invertible. Then

\[
\Phi = \theta_2 \theta_1^{-1}
\]

is a strong symmetry for \( u = K(u) \).

In his beautiful paper [18] Magri has shown that many of the popular soliton equations are bi-Hamiltonian systems (in fact they are, more or less, \( N \)-Hamiltonian systems for arbitrary \( N \) in a sense to be explained later.)

3. Hereditary symmetries admitting a symplectic-implectic factorization.

First let us recall the notion of hereditary symmetry. An operator-valued function \( \Phi(u) : S \to S, u \in M \), is called a hereditary symmetry [10] if \([\Phi'(u), \Phi(u)]\) is a symmetric bilinear operator for all \( u \in M \). By this we mean that

\[
(v, w) \to \Phi'[\Phi v] w - \Phi \Phi'[v] w
\]

is symmetric in \( v, w \in S \). The importance of these operators lies in the fact [10] that if a hereditary symmetry \( \Phi \) is a strong symmetry for \( u = K(u) \)

then it is also a strong symmetry for the equation

\[
u = \Phi(u) K(u).
\]

Now, consider two implectic operators \( \theta_1(u) \) and \( \theta_2(u) \). We call them compatible if \( \theta_1(u) + \theta_2(u) \) is again an implectic operator. A criterion for compatibility is provided by considering the following brackets which are mixed, that is involve both \( \theta_1 \) and \( \theta_2 \)

\[
[a, b, c]_1 = \langle b, \theta_1'[\theta_2 a] c \rangle,
\]

\[
[a, b, c]_2 = \langle b, \theta_2'[\theta_1 a] c \rangle.
\]

A simple calculation yields:

Lemma 1 (see footnote 10 in [18]): \( \theta_1 \) and \( \theta_2 \) are compatible if and only if the bracket \( [a, b, c] = [a, b, c]_1 + [a, b, c]_2 \) satisfies the Jacobi identity.

An immediate consequence of the above is

Remark 4. If \( \theta_1, \theta_2 \) are compatible, then \( \theta_1 + \alpha \theta_2 \) are, for all \( \alpha \in \mathbb{R} \) implectic.

Theorem 1. Let \( \theta_1, \theta_2 \) be compatible implectic operators and assume that \( \theta_2 \) is invertible. Then \( \Phi = \theta_1 \theta_2^{-1} \) is hereditary.

Proof. By \{ \} and \{ \} we denote the brackets with respect to \( \theta_1 \) and \( \theta_2 \), respectively. Consider arbitrary \( b \in S^* \) and \( a, c \in S \) and let \( \vec{b} = \Phi^1 b, \vec{a} = \theta_1^{-1} a, \vec{c} = \theta_2^{-1} c \). We have to prove that

\[
A = \langle b, [\Phi', \Phi](a, c) \rangle - \langle b, [\Phi', \Phi](c, a) \rangle
\]

is equal to zero. So, let us calculate this quantity:

\[
A = \langle b, \theta_1'[\theta_1 \theta_2^{-1} a] \theta_2^{-1} c \rangle - \langle b, \theta_1'[\theta_2 \theta_1^{-1} c] \theta_2^{-1} a \rangle
\]

- \langle b, \theta_1 \theta_2^{-1} \theta_1'[a] \theta_2^{-1} c \rangle + \langle b, \theta_1 \theta_2^{-1} \theta_1'[c] \theta_2^{-1} a \rangle
\]

- \langle b, \theta_1 \theta_2^{-1} \theta_1'[\theta_2 \theta_1^{-1} a] \theta_2^{-1} c \rangle + \langle b, \theta_1 \theta_2^{-1} \theta_1'[\theta_2 \theta_1^{-1} c] \theta_2^{-1} a \rangle
\]

+ \langle b, \theta_1 \theta_2^{-1} \theta_1 \theta_2^{-1} \theta_1'[a] \theta_2^{-1} c \rangle - \langle b, \theta_1 \theta_2^{-1} \theta_1 \theta_2^{-1} \theta_1'[c] \theta_2^{-1} a \rangle.

The first line of this expression is equal to

\[
\{\vec{a}, b, \vec{c}\} + \{\vec{c}, \vec{a}, b\}.
\]

By the Jacobi identity and by the definition of
\[-\{b, c, d\} = \{b, d, c\}. \tag{11a}\]

The second line is equal to
\[|\tilde{a}, \tilde{c}| + |\tilde{c}, \tilde{b}, \tilde{a}|. \tag{11b}\]

and the third line equals
\[-|\tilde{a}, \tilde{b}, \tilde{c}| + |\tilde{c}, \tilde{b}, \tilde{a}|. \tag{11c}\]

Finally, for the fourth line we get
\[\{\tilde{a}, \Phi^*\tilde{b}, \tilde{c}\} - \{\tilde{b}, \Phi^*\tilde{b}, \tilde{a}\}. \tag{11d}\]

Again, by the Jacobi identity and the definition of \(|\ |\) this is shown to be equal to
\[\{\Phi^*\tilde{b}, \tilde{a}, \tilde{c}\} = \{\tilde{a}, \Phi^*\tilde{b}, \tilde{c}\}. \tag{11e}\]

Hence, we obtain
\[A = |\tilde{b}, \tilde{a}, \tilde{c}| + |\tilde{a}, \tilde{c}| + |\tilde{c}, \tilde{b}| + |\tilde{b}, \tilde{a}|. \tag{11f}\]

Then lemma 1 implies \(A = 0\). ■

**Remark 5.**

(i) Actually the proof of the theorem shows a little bit more. If \((\Phi^*)^{-1}\) exists or, if \(\Phi^*S^*\) is (weak-star) dense in \(S^*\) then by virtue of (11e) we can conclude that if \(\Phi\) is hereditary, then \(\theta_1\) and \(\theta_2\) are compatible. Hence if \(\Phi^*S^*\) is always dense in \(S^*\) then \(\theta_1, \theta_2\) are compatible if and only if \(\theta_1\theta_2^{-1}\) is hereditary.

(ii) If in addition to the assumptions of theorem 1 \(\theta_1\) is also invertible, then \(\Phi\) and \(\Phi^{-1}\) are hereditary.

The above theorem makes it desirable to find as many implictic operators as possible, in order to construct hereditary symmetries from the compatible pairs. A systematic method to construct new implictic operators is to use Bäcklund transformations (see section 5). Another method to find new implictic operators is given by:

**Theorem 2.** Under the assumptions of theorem 1 all the operators \(\Phi^n\theta_2, n = 1, 2, \ldots\) are implictic.

**Proof.** To avoid the cumbersome computing of the derivatives involved we give a short proof for a restricted case, namely for the case that \(\theta_1 - \lambda\theta_2\) are invertible for all \(\lambda\) in a neighbourhood of zero.

Since \(\theta_1 - \lambda\theta_2\) are again implictic, the operators \((\theta_1 - \lambda\theta_2)^{-1}\) are symplectic. Since the symplectic operators form a vector space the \(n\)th derivative
\[\frac{d^n}{d\lambda^n} (\theta_1 - \lambda\theta_2)^{-1}, \quad n = 0, 1, \ldots\]

at \(\lambda\) equal to zero must again be symplectic. But this \(n\)th derivative is equal to \(\theta_1^{-1}(\theta_2\theta_1^{-1})^n\). Hence, the inverse of that operator must be implictic. But this inverse is equal to
\[\Phi^n\theta_1 = \Phi^{n+1}\theta_2, \quad n = 0, 1, 2, \ldots\]

4. Motivation and illustrations of the basic notions

In this section we illustrate the notions of a gradient function, a symmetry, a conserved covariant, a strong symmetry, a Noether operator and an implictic operator by using the KdV as an example.

We shall consider the KdV in the form
\[u_t = u_{xxx} + 6uu_x. \]

Then
\[K(u) = u_{xxx} + 6uu_x, \quad K' = D^3 + 6uD + 6u_x. \]
4.1 The gradient of a functional

Recall that the gradient of a functional \( I \) is defined by the equation

\[
I'[v] = \langle \text{grad} \ I, v \rangle,
\]

where \( I'[v] = \partial_\epsilon I(u + \epsilon v) \big|_{\epsilon=0} \) and \( \langle , \rangle \) denotes the relevant scalar product.

**Example 1.**

Let

\[
I = \int \left( -\frac{u^2}{2} + u^3 \right) \, dx.
\]

Then

\[
I'[v] = \int \left( -u_x v_x + 3u^2 v \right) \, dx = \int (u_{xx} + 3u^2) \, dx
\]

= \langle u_{xx} + 3u^2, v \rangle.

Hence

\[
\text{grad} \ I = u_{xx} + 3u^2.
\]

Note that in general if

\[
I = \int \rho(u) \, dx, \quad \rho(u) = \rho(x, u, u_x, \ldots),
\]

then

\[
\text{grad} \ I = \left( \frac{\partial}{\partial u} - D \frac{\partial}{\partial u_x} + D^2 \frac{\partial}{\partial u_{xx}} - \ldots \right) \rho(u).
\]

**Example 2.**

Let

\[
I = \int f(u)g(u) \, dx,
\]

where \( f, g \) are functions of \( x, u, u_x, \ldots \). Then

\[
I'[v] = \langle f'[v], g \rangle + \langle f, g' \rangle
\]

= \langle (f')^*[g], v \rangle + \langle (g')^*[f], v \rangle.

Hence

\[
\text{grad} \ I = (f')^*[g] + (g')^*[f].
\]

It is well known that \( \gamma \) is a gradient function, i.e. there exists a potential \( I \) such that \( \gamma = \text{grad} \ I \), iff \( \gamma' = (\gamma')^* \).

**Example 3.**

Let

\[
\gamma = u_{xx} + 3u^2,
\]

then

\[
\gamma' = D^2 + 6u = (\gamma')^*.
\]

Hence, \( \gamma \) is a gradient function as expected since its potential is the functional \( I \) of example 1.

4.2 Symmetries

For a motivation of the definition of symmetries given here the reader is referred to [6]. It is obvious that the KdV is invariant under translation in \( x \). To this invariance there corresponds (see [6]) the symmetry \( \sigma = u_x \). Let us verify this:

\[
\sigma = u_x, \quad \sigma' = D,
\]

then

\[
K'[\sigma] - \sigma'[K] = (D^3 + 6uD + 6u_x)u_x - D(u_{xxx} + 6uu_x) = 0.
\]

The first few symmetries of the KdV are

\[
\sigma^{(1)} = u_x \quad \text{(invariance under} \ x\text{-translation)}.
\]
The equations

\[ u_i = \sigma^{(i)}, \quad i = \text{integer} \]

define the hierarchy of exactly solvable equations associated with the KdV.

4.3. Conserved covariants

Let us show that if \( \gamma \), in addition to satisfying the equation

\[ \gamma'[K] + (K')^*\gamma = 0, \]

is a gradient function, then its potential is a conserved functional:

For the functional \( I \) is a conserved functional for the equation \( u_i = K \) iff

\[ \frac{dI}{dt} = 0, \]

or

\[ I'[u_i] = \langle \text{grad } I, u_i \rangle = \langle \text{grad } I, K \rangle = 0. \]

Differentiating the last equation along the arbitrary direction \( v \) and defining \( \gamma \) by

\[ \gamma = \text{grad } I, \]

we find

\[ \langle \gamma'[v], K \rangle + \langle \gamma, K'[v] \rangle = 0, \]

or

\[ \langle (\gamma'[K] + (K')^*\gamma), v \rangle = 0, \]

where we have used \( \gamma = (\gamma')^* \) since \( \gamma = \text{grad } I \).

The first few conserved covariants and the corresponding conserved functionals of the KdV are

\[ \gamma^{(1)} = u, \quad I^{(1)} = \int_{-\infty}^{\infty} \frac{u^2}{2} \, dx, \]

\[ \gamma^{(2)} = u_{xx} + 3u^2, \quad I^{(2)} = \int_{-\infty}^{\infty} \left( -\frac{u_x^2}{2} + u^3 \right) \, dx, \]

\[ \gamma^{(3)} = u_{xxxx} + 10uu_{xx} + 5u_x^4 + 10u^3, \quad I^{(3)} = \int_{-\infty}^{\infty} \left( \frac{u_{xx}^2}{2} + \frac{5}{2} u^2 u_{xx} + \frac{10}{4} u^4 \right) \, dx. \]

Example 4.

Verify that \( \gamma^{(2)} \) is a conserved covariant: \( (D^2 + 6u)(u_{xx} + 6uu_x) + (-D^3 - 6Du + 6u_x)(u_{xx} + 3u^2) = 0. \)

4.4 Strong symmetries

A strong symmetry of an equation is an operator which generates from a given symmetry of the equation a new symmetry. The strong symmetry of the KdV is given by

\[ \Phi = D^2 + 4u + 2u_x D^{-1}. \]

Let us verify that the equation

\[ \Phi'[K] - [K', \Phi] = 0 \]

is satisfied:

\[ \Phi'[v] = 4v + 2v_x D^{-1}. \]

Therefore,

\[ \Phi'[K]v = [4K + 2(DK)D^{-1}]v = 4(u_{xxx} + 6uu_x)v + 2(u_{xxxx} + 6u_x^2 + 6uu_x)D^{-1}v, \]

\[ K'[\Phi]v = (D^3 + 6uD + 6u_x)(u_{xx} + 4uv_x + 2u_x D^{-1}v) \]

\[ = v_{xxxx} + 10uv_{xxx} + 20u_v_{xx} \]

\[ + (18u_{xx} + 24u^2)v_x + (10u_{xxx} + 60uu_{xx})v \]

\[ + 2(u_{xxxx} + 6u_x^2 + 6uu_x)D^{-1}v, \]
\[ \Phi K'[u] = (D^2 + 4u + 2u_x D^{-1})(v_{xxx} + 6uv_x + 6u_v) \\
= v_{xxxx} + 10uv_{xxx} + 20u_x v_{xx} \\
+ (18u_{xx} + 24u^2)v_x + (6u_{xxx} + 36uu_x)v. \]

Therefore

\[ \Phi'[K]v = K'[\Phi v] - \Phi K'[v]. \]

Using the expressions for \( \sigma^{(i)} \) given earlier, it can be easily verified that

\[ \sigma^{(i)} = \Phi \sigma^{(i-1)}, \quad i = 1, 2, 3. \]

Similarly,

\[ \gamma^{(i)} = \Phi^{*} \gamma^{(i-1)}, \quad i = 1, 2, 3. \]

The use of the operator \( \Phi \) is well established in the literature, see for example [1].

4.5. Noether operators

The expressions for \( \sigma^{(i)} \) and \( \gamma^{(i)} \) suggest that

\[ \sigma = D\gamma, \]

i.e. that \( \theta^{(i)} = D \) is a Noether operator of the KdV. Let us verify this:

\[
\begin{align*}
(\theta'[K] - \theta(K')^* - K'\theta)v &= 0 \\
- D(-v_{xxx} - 6(uv)_x + 6u_v) \\
- (D^3 + 6uD + 6u_x)v_x &= 0. 
\end{align*}
\]

Actually

\[ \theta^{(2)} = D^3 + 4uD + 2u_x, \]

is also a Noether operator of the KdV, since

\[ \theta^{(2)} = \Phi \theta^{(1)}. \]

4.6 Implictic operators

Let us write the equation \( u_i = K \) in the form

\[ u_i = \theta \text{grad} H, \]

where \( \theta \) is some operator. We will now motivate the properties that \( \theta \) must have in order for the above equation to be in a Hamiltonian form. Recall that in classical mechanics the Poisson bracket is defined in such a way that if \( H \) is the Hamiltonian and \( I \) some other functional, then

\[ \{I, H\} = I_i. \]

In this case

\[
\{I, H\} = I_i = I'[u_i] = \langle \text{grad} ~ I, u_i \rangle = \langle \text{grad} I, \theta \text{grad} H \rangle. 
\]

Therefore, a natural Poisson bracket is defined by

\[ \{I, H\} \equiv \langle \text{grad} I, \theta \text{grad} H \rangle. \quad (12a) \]

A Poisson bracket usually has two properties: It is skew symmetric and it satisfies the Jacobi identity. The skew symmetry implies

\[ \theta^{*} = - \theta. \]

We shall show that the Jacobi identity implies

\[ \langle a, \theta'[\theta b]c \rangle + \langle b, \theta'[\theta c]a \rangle + \langle c, \theta'[\theta a]b \rangle = 0, \]

where \( a, b, c \) are gradient functions:

For

The Jacobi identity

\[ \{(A, B), C\} + \{(B, C), A\} + \{(C, A), B\} = 0, \]

implies

\[ \{(a, \theta b), C\} + \{(b, \theta c), A\} + \{(c, \theta a), B\} = 0, \quad (12b) \]

where \( a, b \) and \( c \) are the gradients of \( A, B, C \) (and therefore \( a', b', c' \) are symmetric). In order to proceed we need to evaluate the gradient of say \( \langle a, \theta b \rangle \). So let us consider

\[ I \equiv \langle f, \theta g \rangle; \quad \theta^{*} = - \theta. \]
Then
\[ I'[v] = \langle f'[v], \theta g \rangle + \langle f, \theta'[v]g \rangle + \langle f, \theta g'[v] \rangle \]
\[ = \langle f^*\theta g, v \rangle - \langle (g')^*\theta f, v \rangle + \langle f, \theta'[v]g \rangle. \]

But \( \theta'[v]g \) is an operator depending on \( g \) and acting linearly on \( v \), i.e.
\[ \theta'[v]g \equiv L(g)v. \]

Hence
\[ \text{grad} \langle f, \theta g \rangle = (f')^*\theta g - (g')^*\theta f + L^*(g)f. \] (12c)

Using (12c) and (12a) in (12b), we obtain after cancellations
\[ \langle L^*(b)a, \theta c \rangle + \langle L^*(c)b, \theta a \rangle + \langle L^*(a)c, \theta b \rangle = 0, \]
\[ \text{or} \]
\[ \langle a, L(b)\theta c \rangle + \langle b, L(c)\theta a \rangle + \langle c, L(a)\theta b \rangle = 0, \]
\[ \text{or} \]
\[ \langle a, \theta'[\theta c]b \rangle + \langle b, \theta'[\theta a]c \rangle + \langle c, \theta'[\theta b]a \rangle = 0. \]

**Example 5.**
The KdV can be written in the form
\[ u_t = D(u_{xx} + 3u^2). \]

Hence since \( D \) is symplectic (since it is skew-symmetric and also it trivially satisfies (7)) and \( u_{xx} + 3u^2 \) is a gradient function (the gradient of \( I^{(0)} \)), the above form defines a Hamiltonian system.

**Example 6.**
The KdV can also be written in the form
\[ u_t = (D^3 + 4uD + 2u_x)u = \theta^{(0)}u. \]

The function \( u \) is obviously a gradient function (the gradient of \( I^{(0)} \)), Hence, the above equation defines another Hamiltonian system iff the operator \( \theta^{(0)} \) is symplectic (i.e. if it satisfies (7)). It is in general cumbersome to check that an operator like \( \theta^{(0)} \) satisfies (7). However, using the results of this paper we usually can avoid the direct calculation (see the next section).

5. Bäcklund transformations for symplectic operators

For Bäcklund transformations the reader is referred to [5] or [22]. To make this paper self contained, we recall the definition of a Bäcklund transformation.

A function \( B(u, s) \) in the two arguments \( u \in M_1, s \in M_2 \) with values in a third vector space \( S_3 \) is called admissible if the implicit function given by \( B(u, s) = 0 \) gives rise to a one-to-one map between the corresponding tangent spaces, i.e. we require that for \( B = 0 \) the linear maps \( B_u \) and \( B_s \) from \( S_1 \) to \( S_1 \) and \( S_2 \) to \( S_2 \) respectively are invertible. Here \( B_u \) and \( B_s \) denote the partial derivatives with respect to \( u \) and \( s \).

An admissible function \( B(u, s) \) (or rather the implicit function between \( u \) and \( s \) given by \( B = 0 \)) is said to be a Bäcklund transformation between the evolution equations

\[ u_t = K(u), \quad u(t) \in M_1, \] (13a)
\[ s_t = G(u), \quad s(t) \in M_2, \] (13b)

if, for all \( t \),

\[ B(u(t), s(t)) = 0, \quad \text{whenever } B(u(0), s(0)) = 0. \]

In the following we always assume that \( B(u, s) \) is admissible.

On the context of transformations of operators an important role is played by the operator \( T: S_1 \to S_2 \) given by

\[ T = B_s^{-1}B_u. \] (14)
Of course, \( T \) depend on \( u \) and \( s \), where \( u \) and \( s \) are related by \( B(u, s) = 0 \). Let \( d_t \) and \( d_u \) denote total derivatives with respect to \( s \) and \( u \). Then \( d_T \) has a useful symmetry property expressed by the formula

\[
(d_T)[v]w = (d_T)[Tw]T^{-1}v, \quad v \in S_2, \ w \in S_1.
\]

(15)

For the proof see [7, lemma]. This formula immediately yields a similar property for \( T^*: S^! \to S^\dagger \) expressed by

\[
(d_T^*[v]a, w) = (d_T)[Tw]^T[a, T^{-1}v]
\]

(16)

for arbitrary \( v \in S_2 \), \( w \in S_1 \) and \( a \in S^\dagger \).

**Theorem 3.** For \( s \in M \) define operators

\[
\Omega(s): S^\dagger \to S^\dagger
\]

by

\[
\Omega(s) = T\theta(u)T^*;
\]

(17)

where \( \theta(u), u \in M \), are given operators \( S^\dagger \to S^\dagger \). Then \( \Omega \) is symplectic if and only if \( \theta \) is symplectic.

**Proof.** It is obvious that the skew-symmetry is preserved by the transformation (17). Hence, we only need to prove that the Jacobi identity for \( \{ \} \) (with respect to \( \Omega \)) follows from the Jacobi identity for \( \theta \). To do that consider arbitrary \( a, b, c \in S^\dagger \), define \( \tilde{a} = T^*a \), \( \tilde{b} = T^*b \), \( \tilde{c} = T^*c \) and denote by \( \langle b, (d_T)[\Omega a]T^*c \rangle \) the quantity \( \langle b, (d_T)(\Omega a)T^*c \rangle \). Elementary differential calculus then yields:

\[
\{ a, b, c \} = \langle b, (d_T)[\Omega a]T^*c \rangle
\]

(18)

Note that since \( s \) is related to \( u \) by \( B = 0 \) the total derivative of \( \theta(u) \) is given by

\[
(d_T)[v] = -\theta_u[T^{-1}v].
\]

(19)

Inserting this in the above equation we obtain

\[
\{ a, b, c \} = \langle b, (d_T)[\Omega a]T^*c \rangle
\]

for arbitrary \( v \in S_2 \), \( w \in S_1 \) and \( a \in S^\dagger \).

Hence, the Jacobi identity for \( \{ \} \) is clearly equivalent to that for \( \{ \} \). \( \blacksquare \)

This result suggests that a similar transformation formula should exist for Noether operators in general. This is expressed by

**Theorem 4.** Consider a Bäcklund transformation \( B = 0 \) between (13a) and (13b) and define

\[
\Omega(s) = T\theta(u)T^*.
\]

Then \( \Omega \) is a Noether operator for (13b) if and only if \( \theta \) is a Noether operator for (13a).

**Proof.** Again, it suffices to prove one direction.

Apart from formulas (14), (16) and (18) we are going to need:

\[
\Omega = -TK, \quad d_sK = -K_uT^{-1}.
\]

(20)

(21)

Eq. (20) is an immediate consequence of the fact that \( B = 0 \) is a Bäcklund transformation between the corresponding equations, and (21) is valid for the same reasons as (18).

Using (20) and (21) we get for the total derivative of \( G \):

\[
d_sG[\cdot] = -(d_T)[\cdot]K + TK_u[T^{-1} \cdot].
\]

(22)

Now, let us calculate the crucial quantity

\[
\Omega_G = \Omega G^\dagger - G, \Omega
\]

\[
= (d_T)[G]T^* + T(d, \theta)[G]T^* + T\theta(d, T^*)[G]
\]

\[
+ \Omega(d, T)[\cdot]K - \Omega(TK_uT^{-1})^* + (d_T)[\Omega \cdot]K - TK_uT^{-1}\Omega.
\]
Now, using (15) and (20) one sees that the first and the sixth terms cancel. For the same reason the third and the fourth terms cancel, and the remaining terms can be written in the following form:

$$\Omega_t[G] - \Omega G^\pi - G_\pi$$

$$= T(\theta_\pi[K] - \theta K^\pi - K_\pi)T^*.$$  \hspace{1cm} (23)

Hence, if $\theta$ is a Noether operator $\Omega$ must be one (and vice versa).

**Remark 6.**

(i) One should observe that the transformation (17) preserves the compatibility of symplectic operators. As we shall see this is a very useful property.

(ii) Clearly theorem 1 and theorem 4 yield a Bäcklund transformation for those hereditary symmetries which admit a symplectic-implectic factorization. But—as we have shown in [7]—this transformation carries over even to the case were the hereditary symmetry can not be factorized.

(iii) Obviously, there exists a similar transformation formula for symplectic operators (and inverse-Noether operators). This formula is easily guessed, it is

$$\tilde{J}(s) = T^{-1}\tilde{J}(u)T^{-1}.$$ \hspace{1cm} (24)

**6. Examples and applications**

**6.1. Some implectic operators**

Let $S$ be the space of $C^n$-functions on $\mathbb{R}$ rapidly vanishing at $\pm \infty$. $D$ denotes differentiation with respect to $x \in \mathbb{R}$, $D^{-1}$ denotes its inverse given by

$$f(x) \mapsto (D^{-1}f)(x) = \int_\infty^x f(\xi) \, d\xi.$$  \hspace{1cm}

If we take $S^* = \{D^{-1}s|s \in S\}$, with a bilinear form given by

$$\langle s_1, s_2 \rangle = \int_\infty^x s_1(x)s_2(x) \, dx,$$  \hspace{1cm}

then $D: S^* \rightarrow S$ and $D^{-1}: S \rightarrow S^*$ are skew-symmetric. One easily proves that the following operators are implectic:

$$\theta_1(u) = \alpha D + \phi(u)D\phi(u),$$ \hspace{1cm} (25a)

$$\theta_2(u) = \beta D + \phi(u)D + D\phi(u),$$ \hspace{1cm} (25b)

where $u \in \mathcal{M}$: $\alpha$, $\beta$ are arbitrary numbers and $\phi$ any $C^n$-function.

Considering Bäcklund transformations and theorem 3 one can generate out of these implectic operators new ones. For example

$$B(u, s) = s + \alpha u + \beta u^2 + u_x = 0$$

transforms $\theta(u) = D$ into

$$\Omega(s) = (\alpha + 2\beta u + D) \, D(\alpha + 2\beta u - D)$$

$$= \alpha^2 D - D^3 - 2\beta(Ds + sD).$$

Hence, (renaming the above operator) we have found a new family of implectic operators:

$$\theta_3(u) = \alpha D + \beta D^3 + \gamma(Du + uD).$$ \hspace{1cm} (25c)

An other implectic operator is given by

$$\theta(u) = DuD^{-1}uD.$$  \hspace{1cm}

Using lemma 1 one easily finds that this operator is compatible with $\alpha D + \beta D^3$. Hence

$$\theta_4(u) = \alpha D + \beta D^3 + \delta DuD^{-1}uD$$ \hspace{1cm} (25d)

is again a family of implectic operators.
6.2. Construction of hereditary symmetries and more implectic operators

Since the parameters $\alpha, \beta, \gamma, \delta$ in (25) are arbitrary, it follows that the corresponding parts of operators constitute compatible pairs of implectic operators. Using this trivial fact one gets (using theorem 1) many hereditary symmetries. For example:

\[ \Phi_1(u) = \alpha + \phi(u)D\phi(u)D^{-1}, \quad (26a) \]
\[ \Phi_2(u) = \alpha + \phi(u) + D\phi D^{-1} \quad (26b) \]
\[ \Phi_3(u) = \alpha + \beta D^2 + \gamma(DuD^{-1} + u), \quad (26c) \]
\[ \Phi_4(u) = \alpha + \beta D^2 + \delta DuD^{-1}u, \quad (26d) \]

and

\[ \Phi_5(u) = (\alpha + \beta D^2 + \gamma(DuD^{-1} + u))(1 - D^2)^{-1}, \quad (26e) \]
\[ \Phi_6(u) = (\alpha + \beta D^2 + \delta DuD^{-1}u)(1 - D^2)^{-1}, \quad (26f) \]

and so forth.

In the cases $\Phi_1$ to $\Phi_4$ the role of $\theta_2$ (in theorem 1) was played by $D$ and in the cases $\Phi_5$ and $\Phi_6$ by $(D - D^3) = D(1 - D^2)$.

One can now use theorem 2 to obtain from $\Phi_1, \ldots, \Phi_6$ new implectic operators, namely

\[ \theta_{nk}(u) = (\Phi_k(u))^nD; \quad k = 1, \ldots, 4 \text{ and } n \in \mathbb{N}. \quad (27a) \]

and

\[ \tilde{\theta}_{nk}(u) = (\Phi_k(u))^n(D - D^3); \quad k = 5, 6 \text{ and } n \in \mathbb{N}. \quad (27b) \]

Theorem 1 also indicates that if one changes the spaces $S$ and $S^\ast$ in a suitable way, so that the $\Phi_i^{-1}$ make sense, then these inverse operators are also hereditary (this follows from interchanging the role of $\theta_1$ and $\theta_2$ in theorem 1).

At this point we emphasize that not all hereditary symmetries admit a symplectic decomposition. A counterexample is

\[ \Phi(u) = D + DuD^{-1}, \]

which is the hereditary symmetry of Burgers equation ([10] or [19]).

Of course, a suitable application of our results yields far more hereditary symmetries (and implectic operators) than we have given so far. But instead of trying to be complete in this respect, we give a brief outline of how these results can be applied in the theory of nonlinear evolution equations.

6.3. The corresponding Hamiltonian systems, their symmetries, conservation laws and soliton solutions

We assume that $\Phi(u)$ is any of the hereditary symmetries given by (26). Since $\Phi(u)$ is invariant under $x$-translation it follows that

\[ \Phi[u_x] = [D, \Phi]. \quad (28) \]

Hence $\Phi$ is a strong symmetry for the evolution equation:

\[ u_t = u_x. \quad (29) \]

So, the basic property [10] of hereditary symmetries implies that $\Phi$ is also a strong symmetry for any of the following evolution equations:

\[ u_t = K_n(u), \quad n = 0, 1, 2, \ldots, \quad (30a) \]

where

\[ K_n(u) = \Phi(u)^n u_x. \quad (30b) \]

If $\Phi^{-1}$ makes sense then $\Phi$ is also a strong symmetry for the inverse equations:

\[ (\Phi(u))^n u_t = u_x, \quad n = 1, 2, \ldots, \quad (31a) \]
This follows from the fact that $\Phi^{-1}$ is then also hereditary and that (31a) can be written in the form

$$u_t = K_n(u) = (\Phi(u))^{-n}u_x, \quad n = 1, 2, \ldots \quad (31b)$$

Among these equations are

(i) the Korteweg–de Vries equation

$$u_t = u_{xxx} + 6uu_x,$$

which is a special case of $u_t = \Phi(u)u_x (\alpha = 0, \beta = 1, \gamma = 2)$;

(ii) the modified KdV

$$u_t = u_{xxx} + 6u_x u^2,$$

which is a special case of $u_t = \Phi(u)u_x (\alpha = 0, \beta = 1, \delta = 4)$;

(iii) the sine-Gordon equation

$$u_t = \frac{1}{2} \sin(2D^{-1}u),$$

which is a special case of $u_t = \Phi(u)u_x (\alpha = 0, \beta = 1, \delta = 4)$.

But among those are also very many other nonlinear evolution equations which have not been considered before. For example:

$$u_t = (\alpha + \beta D^3 + \delta DuD^{-1}u)(1 - D^2)^{-1}u_x. \quad (32)$$

(One should note that the operator $(D^2 - 1)^{-1}$ is an honest operator $S \to S$, namely the one given by

$$u \to e^sD_x^{-1}e^{2s}D^{-1}e^su(x),$$

where $D_x$ are defined to be

$$(D_xv)(x) = \int_{-\infty}^{x} v(\xi) \, d\xi.$$}

We claim that we can provide a complete description of the symmetries, the conserved covariants, the Hamiltonian structure, the conservation laws and the soliton solutions for all these equations.

### 6.3.1. The symmetries

The operator $\Phi(u)$ is a strong symmetry for all the equations given by (30). Now, since $u_x$ is a symmetry for these equations (corresponding to the invariance of the equation under $x$-translation) all the functions $K_n(u)$, $n = 0, 1, 2, \ldots$ (and for $n = -1, -2, \ldots$ if that makes sense) must by symmetries for any of these equations. In other words, all the flows given by the $K_n(u)$, $n \in \mathbb{Z}$, commute.

### 6.3.2. The Noether operators and the fact that the conserved covariants are gradients

The following proposition is very useful for the discussion of equations (30), where $\Phi$ is defined by (26).

**Proposition 2.** Assume that the hereditary symmetry $\Phi$ (which is invariant under $x$-translation) admits the implectic-symplectic factorization

$$\Phi = \theta_1 \theta_2^{-1}. \quad (33)$$

Assume further that

$$G_0 \doteq \theta_2^{-1}u_x \quad (34)_0$$

is a gradient function. Then the equation (30)

$$u_t = K_n(u) = \Phi^n u_x$$

is a bi-Hamiltonian system having $\theta_1$, $\theta_2$, as Noether operators. Furthermore, all the functions

$$G_m \doteq \theta_2^{-1}K_m, \quad m = 1, 2, \ldots \quad (34)_m$$

are conserved covariants of equation (30) and also are gradient functions.
Proof. Theorem 2 implies that all \( \Phi^n \theta_2, m = 1, 2, \ldots \) are implectic. Also, eq. (30) can be written in the form
\[
u_i = \Phi^m \theta_2 G_0 . \tag{30c}\]

Hence, since \( \Phi^m \theta_2 \) is implectic and \( G_0 \) is gradient, proposition 1 implies that (30) is a Hamiltonian system and has \( \Phi^m \theta_2 \) as a Noether operator. We further claim that it also has \( \theta_1 \) and \( \theta_2 \) as Noether operators. Let us prove that for the restricted case when \( \Phi \) is invertible (or at least injective). In this case \( \Phi^m \) is a strong symmetry, hence using remark 2(iii) it follows that \( \theta_2 \) is a Noether operator. But then \( \theta_1 = \Phi \theta_2 \) is also Noether for the same reason. It now trivially follows (since \( K_m \) are symmetries) that \( G_m, m \) as defined by (34), are conserved covariants. Furthermore, (30) can be written in the form
\[
u_i = (\Phi^{m-n} \theta_2) G_m , \tag{30d}\]

and since \( \Phi^{m-n} \theta_2 \) is Noether (for the same reason that \( \theta_2 \) is Noether) and implectic it follows from proposition 1(iii) that all the \( G_m \) are gradient functions. In the case that \( \Phi \) is injective one can directly prove that if \( G_n \) is defined by (34) where \( \theta_2 \) is a Noether operator of (30) and also implectic, then \( G_n \) is a gradient function.

Note that (30d) also indicates that (if \( \Phi \) is invertible) eq. (30) is \( \mathcal{N} \)-Hamiltonian (for arbitrary \( \mathcal{N} \)).

Now let us return to eq. (30), where \( \Phi \) is defined by (26). Then \( \theta_2 \) is either equal to \( D \) (for \( \Phi_1, \ldots, \Phi_4 \)) or equal to \( (D - D^3) \) (for \( \Phi_5 \) and \( \Phi_6 \)). Hence, the corresponding \( G_0 \) are clearly gradients (actually they are the gradients of \( 1(u, u) \) and of \( 1(u, (1 - D^3)^{-1} u) \) respectively). Therefore, the above proposition immediately implies that all \( G_m \) are conserved covariants and gradients.

6.3.3. An orthogonality property

Let \( G_m \) be defined by (34) (where \( \theta_2 \) is skew symmetric but we do not assume that it is Noether). Then using
\[
\theta_2^{-1} \Phi = \Phi^* \theta_2^{-1} \tag{35}\]

we obtain
\[
G_m(u) = (\Phi^*(u))^n G_0 \tag{36}\]

Now, we proceed as in [10] to prove that, for all \( n, m \), we have the orthogonality
\[
\langle G_m, K_n \rangle = 0 . \tag{37}\]

Using (34) and (35) we obtain
\[
\langle G_m, K_n \rangle = \langle G_0, \Phi^{n+m} u \rangle = - \langle u, \theta_2^{-1} \Phi^{n+m} u \rangle = - \langle u, (\Phi^*)^m \theta_2^{-1} \Phi^m u \rangle = - \langle G_m, K_n \rangle .
\]

From the first and last terms of the above
\[
\langle G_m, K_n \rangle = - \langle G_m, K_n \rangle ,
\]

it follows the desired result.

Actually, one could use the above to prove that if \( G_0 \) is a gradient, it is also a conserved covariant for all equations (30) and then conclude from (36) that all \( G_m \) are also conserved covariants (remark 1(iv)).

6.3.4. The conservation laws

It was shown in 6.3.2. that the \( G_m(u) \), \( m = 0, 1, 2, \ldots \), are gradient functions and conserved covariants for all equations (30). Hence, formula (2) implies that all the
\[
P_n(u) = \int_0^1 \langle G_n(\lambda u), u \rangle d\lambda \tag{38}\]

are conserved quantities for these evolution equations.

For the description of soliton solutions the reader is referred to [10]. The procedure suggested in that paper applies equally well to eq. (30).
6.4. The complex case

In section 5.1 we did not specify if the elements of $S$ are real or complex functions. In fact, all the results are valid for both cases. In this subsection we mean by $S$ the space of complex functions and we want to deal with evolution equations which involve absolute values of $u$.

For that situation it turns out ([10], also [18] and [23]) to be convenient to define the bilinear map which connects $S^*$ and $S$ in a different way, namely by

$$
\langle v, u \rangle = \frac{1}{2} \int_{-\infty}^{+\infty} (\bar{v}(x)u(x) + v(x)\bar{u}(x)) \, dx .
$$

In order that this is really a bilinear map, we have to restrict the scalars to the real numbers. Multiplication with imaginary numbers constitutes then an operator (a skew-symmetric one).

A simple calculation yields that then

$$
\theta_1 = i ,
$$

$$
\theta_2 = D ,
$$

$$
\theta_3 = iuD^{-1} \text{Re}(i\bar{u} \cdot) .
$$

are implictic operators. (By $\text{Re}(a \cdot)$ we understand the operator $v \mapsto \text{real part } (av) = \frac{1}{2}(av + \bar{a}\bar{v})$.) All these operators are compatible (lemma 1). This means that

$$
\theta(u) = \alpha i + \beta D + \gamma iuD^{-1} \text{Re}(i\bar{u} \cdot) ; \quad \alpha, \beta, \gamma \in \mathbb{R} ,
$$

constitutes a family of implictic operators which can be used for the construction of hereditary symmetries. A special hereditary symmetry which can be constructed out of (41) is the one found in [10],

$$
\Phi_1(u) = -iD + i4uD^{-1} \text{Re}(\bar{u} \cdot) .
$$

This operator is a special case of

$$
\Phi(u) = -\theta(u) i^{-1} = \theta(u) i \quad = \alpha + \beta iD - \gamma iuD^{-1} \text{Re}(\bar{u} \cdot) .
$$

In fact, all of the analysis given in 6.3. holds for this hereditary symmetry. Among the equations generated by the above $\Phi$ one finds the celebrated Zakharov-Shabat equation [23].

6.5. A new class of exactly solvable third order equations

Let $\alpha$ be a constant parameter and $b(u)$ be any solution of

$$
\frac{d^3b}{du^3} + 8\alpha \frac{db}{du} = 0 .
$$

Then it was shown in [6] that the equation

$$
\frac{d^3b}{du^3} + \alpha u_x^2 + b(u)u_x
$$

is exactly solvable in the sense that it can be written in the form

$$
\frac{d^3b}{du^3} + \Phi(u)u_x,
$$

where $\Phi(u)$ is a hereditary symmetry defined by

$$
\Phi(u) = D^2 + 2\alpha u_x^2 + \frac{2}{3}b
$$

$$
- 2\alpha u_xD^{-1}u_x + \frac{u_x}{3} D^{-1} \frac{db}{du} .
$$

Clearly eq. (43) contains the Gardner equation (linear combination of KdV and modified KdV) as special case when $\alpha = 0$. However, for $\alpha \neq 0$ eqs. (43) yield a new class of exactly solvable equations, namely:

$$
\frac{d^3u}{dt^3} + \alpha u_x^2 + (\tau_1 e^{2\nu t} + \tau_2 e^{-2\nu t} + \tau_3),
$$

$$
\nu = (-2\alpha)^{1/2} ,
$$

where $\tau_1$, $\tau_2$ are constant parameters. Eq. (45)
can be mapped to the modified KdV
\[ s_t = s_{xxx} + \left\{ 3\alpha s^2 + \tau_3 - \frac{(\tau_1 \tau_2)^{1/2}}{3\alpha} \right\} s_x \]  
(46)

through the Bäcklund transformation [6]
\[ B(u, s) = u_x + s + \left( \frac{\tau_1}{3\alpha} \right)^{1/2} e^{su} + \left( \frac{\tau_2}{3\alpha} \right)^{1/2} e^{-su}. \]  
(47)

The Hamiltonian formulation of (46) is well known (see also 6.3. of this paper), but that of eqn. (45) is by no means obvious. However, using the results of sections 3 and 5 we can easily obtain the bi-Hamiltonian formulation of (45):

It turns out that for this example it is more convenient to work with inverse-Noether operators \( J(u) \) instead of Noether operators \( \theta(u) \). Obviously, \( D^{-1} \) is an inverse Noether operator of (46). Hence, from (17) we obtain an inverse-Noether operator of (45)
\[ J(u) = T^* D^{-1} T, \]

\[ T = D + \nu \left( \frac{\tau_1}{3\alpha} \right)^{1/2} e^{su} - \nu \left( \frac{\tau_2}{3\alpha} \right)^{1/2} e^{-su}. \]

Writing that down explicitly we get
\[ J(u) = \frac{2}{3}(\tau_1 \tau_2)^{1/2} J_1(u) - J_2(u), \]  
(48a)

where
\[ J_1(u) = e^{su} D^{-1} e^{-su} + e^{-su} D^{-1} e^{su}, \]  
(48b)
\[ J_2(u) = D + \frac{2}{3} \tau_1 e^{su} D^{-1} e^{su} + \frac{2}{3} \tau_2 e^{-su} D^{-1} e^{-su}. \]  
(48c)

Now, one observes that the hereditary symmetry \( \Phi(u) \) given by (44b) admits the decomposition
\[ J_1(u) \Phi(u) = 2J_2(u) \quad \text{or} \quad \Phi(u) = 2J_1(u)^{-1} J_2(u). \]  
(49)

Since the constants \( \tau_1, \tau_2, \alpha \) are arbitrary it follows that not only \( J(u) \) is symplectic, but that \( J_1(u) \) and \( J_2(u) \) are symplectic as well. Then by remark 5(i), the hereditaryness of \( \Phi(u) \) and (49) we conclude that \( \theta_1 = J_1^{-1} \) and \( \theta_2 = J_2^{-1} \) constitute a compatible pair of symplectic operators. Hence, theorem 2 implies that all the
\[ \Phi(u)^* J_2(u)^{-1} \]
are symplectic.

Using the representation (44a) we can rewrite (45) in the form
\[ u_t = K(u) = \Phi u_x = 2J_1^{-1} G_0, \]  
(45b)

where
\[ G_0 = J_2 u_x = u_{xx} + \frac{2}{3} \tau_1 e^{2su} - \frac{2}{3} \tau_2 e^{-2su}. \]

Further, \( G_0 \) is the gradient of
\[ \int_{-\alpha}^{\alpha} \left\{ \frac{u_x^2}{2} + \frac{\tau_1}{3\nu^2} (e^{2su} - 1) + \frac{\tau_2}{3\nu^2} (e^{-2su} - 1) \right\} dx. \]

(Note, that here we are working again with the bilinear form of section 6.1, only \( S^* \) has been replaced by the space of \( C^\infty \)-functions with at most polynomial growth.) Hence, proposition 2 implies that eq. (45) is a bi-Hamiltonian system with \( \theta_1 = J_1^{-1}, \theta_2 = J_2^{-1} \) as Noether operators. Furthermore, all \( K_m \), \( G_m \) (as defined by (30b) and (34), respectively, where \( \Phi \) is given by (44b)) are symmetries and conserved covariants of eqs. (45). Also all \( G_m \) are gradients and hence they define (see (38)) conserved quantities.

6.6 Some fifth order equations

The following fifth order equations are known to be exactly solvable:
\[ u_t = u_{xxxx} + 5\alpha uu_{xxx} + 10\alpha u_x u_{xx} + \frac{15}{2} \alpha^2 u^2 u_x, \]  
(50a)
\[ u_t = u_{xxxx} + \frac{5}{2} \alpha uu_{xxx} + \frac{5}{2} \alpha u_x u_{xx} + \frac{5}{2} \alpha^2 u^2 u_x, \]  
(50b)
u_t = u_{xxxx} + 10\alpha u u_{xx} + 25\alpha u_x u_x + 20\alpha^2 u_x u_x.

(50c)

Eq. (50a), sometimes called the Lax equation, belongs to the KdV hierarchy [15]. Eqs. (50b) and (50c) were introduced in [4] and [14], respectively.

Eq. (50a) can be written in the form

u_t = \Phi^2 u_x,

(51a)

\Phi = D^2 + \alpha(DuD^{-1} + u),

(51b)

where \Phi is the hereditary symmetry of the KdV equation (see (26c) with \alpha = 0, \beta = 1). Hence its Hamiltonian formulation follows from that of the KdV. For example \theta = D and

\theta = \Phi D = D^3 + \alpha(Du + uD) \tag{52}

are Noether operators of eq. (50a). But what is the Hamiltonian formulation of eqs. (50b) and (50c)? (It can be easily seen that these equations do not admit D as an Noether operator.)

We claim that \theta as defined by (52) is also a Noether operator for eqs. (50b) and (50c). This follows from the fact that eqs. (50) can be written in the form

u_t = [D^3 + \alpha(Du + uD)] \text{grad} \int_{-\infty}^{\infty} \left( -\frac{u_x^2}{2} + \frac{\beta u_x^3}{3} \right) dx, \tag{53}

where eqs. (50a), (50b) and (50c) correspond to \beta = \alpha/4, \beta = 3\alpha/2, \beta = 4\alpha, respectively, and \text{grad} \text{H} denotes the gradient of the functional H.

Eq. (53) provides the common Hamiltonian structure of eqs. (50) and (then proposition 1 implies that) \theta is their common Noether operator.

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References


Note added in proof
Since this paper was submitted we learned about other work dealing with, what we call, hereditary symmetries. The principal contributions are:
The connection to our work is discussed in:
B. Fuchssteiner, Progress of Theoretical Physics, vol. 65 (3) 1981, where also some of the results of this paper are extended.